Decision Support

An efficient and simple model for multiple criteria supplier selection problem

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Abstract

Simply looking for vendors offering the lowest prices is not “efficient sourcing” any more. Selection of suppliers is a multiple criteria decision. We propose a weighted linear program for the multi-criteria supplier selection problem. In addition to mathematical formulation, this paper studies a transformation technique which enables our proposed model to be solved without an optimizer. The model for multi-criteria supplier selection problem can be easily implemented with a spreadsheet package. The model can be widely applied to practical situations and does not require the user with any optimization background.
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1. Introduction

Competitive advantages associated with supply chain management (SCM) philosophy can be achieved by strategic collaboration with suppliers and service providers. The success of a supply chain is highly dependent on selection of good suppliers. Simply looking for vendors offering the lowest prices is not “efficient sourcing” any more. Multiple criteria need to be taken into account when selecting suppliers. The supplier selection problem has received considerable attention in academic research and literature. Early in 1960s, Dickson identified 23 criteria that ought to be considered by purchasing personnel in evaluating suppliers [5]. A latter review by Weber et al. [11] reported that well over half of 74 research papers reviewed addressed the supplier selection problem with multiple criteria. Another comprehensive review by De Boer et al. [4] discussed a framework for supplier selection. The framework covers different phases of the supplier selection process, including pre-qualification, formulation of criteria, final evaluation, etc. In the final evaluation phase of suppliers, after pre-qualification, quantitative models incorporating multi-criteria were constructed. These models

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are based on multi-objective optimization (MOP) [3,12,13], data envelopment analysis (DEA) [7,9,10,12,13], analytic hierarchical process (AHP) [1,2,6] and simple multi-attribute rating technique (SMART) [9].

These models provide systematic approaches for purchasing managers to evaluate and score suppliers with multi-criteria. Nevertheless, these models are not easy to implement. Models based on multi-objective optimization require the decision makers to exogenously specify the exact values of weights of individual criteria. It is however difficult to obtain precise weight values. The weight determination is a challenging task for implementing the MOP approach. A similar problem is faced when decision makers choose the SMART approach. On the other hand, to assist decision makers in determination of the weights, the AHP approach provides for interactive comparisons for users to obtain the weights. Decision makers are required to perform pair-wise comparisons between the criteria and the supplier alternatives under a particular criterion. However, the results are highly dependent on the subjective judgments of the decision makers. Decision makers have to specify not only the direction of relative importance (e.g., Criterion A is more important than Criterion B) but also the degree of the relativity (e.g., Criterion A is extremely/very strongly more important than Criterion B). The requirement of user preference is too demanding, when implementing these models.

DEA appears to be the easiest for practical implementation. The DEA approach does not require the decision maker to pre-define the weights. Weights are endogenously determined when solving a DEA model. DEA can automatically derive optimal weights of criteria with the performance scores of the suppliers. The solutions of DEA models require a linear optimizer, which is available to a decision maker. An individual DEA model is required to be optimized for each supplier. In DEA models applied to supplier selection problems [7,9,10,12,13], decision makers cannot have any involvement or control for the importance of the criteria. To some extent, these DEA approaches are black-box models for decision makers in real situations.

There are many real situations where the decision makers are able to tell the criteria importance ranking (although they cannot tell the exact values of weights). The decision makers may not have enough knowledge to assign exact weight values but they can rank the importance by their expertise or experience. In this kind of decision-making environment, the two abovementioned streams of approaches (weights determined exogenously and weights determined endogenously) may not be applicable. We would like to propose an alternative mathematical model for the multi-criteria supplier selection problem. Our models retain the advantage of DEA, that requires no pre-defined weight values. At the same time, our model can incorporate some user control by ranking of relative importance of the criteria. Unlike AHP or MOP models, decision makers only rank the relative importance of criteria, rather than specifying the degree of relativity. This sort of subjective judgments is much less demanding. Moreover, our proposed model does not require users with any optimization background. The model can be implemented with the commonly available spreadsheet package and can be widely applied to practical situations in industries.

The organization of this paper is as follows. The mathematical program model based on a multi-criteria decision making method is presented in Section 2. A transformation technique is briefly presented in Section 3. The transformation enables our proposed model to be solved without any optimization procedure. Section 4 analyzes the sensitivity of supplier scores when the order of criteria ranking changes and when value of a measure changes. The sensitivity analysis offers an important tool in practical implementation while negotiating with suppliers. Illustrative examples and comparisons with other approaches are given in Section 5. Some concluding remarks are provided in Section 6.

2. Mathematical formulation

Inspired by a multi-criteria decision method [8], we propose a weighted linear model for the supplier selection problem with multiple criteria. We consider a situation in which a set of $I$ suppliers is available for a company. The purchasing manager would like to evaluate these suppliers based on $J$ criteria. We evaluate a supplier $i$ ($i = 1, 2, 3, \ldots, I$) by converting multiple measures under all criteria into a single score $S_i$. The measure of supplier $i$ under criteria $j$ is denoted as $x_{ij}$ ($i = 1, 2, 3, \ldots, I$, $j = 1, 2, 3, \ldots, J$). We assume all measures are positively related to the score of a supplier. If there is a negatively related criterion, transformation of negativity or taking reciprocal can be applied for conversions. A common scale for all measures is also an important issue. A particular criterion measure, in a large scale, may always dominate the score. For this, we propose normalizing all measures $x_{ij}$ into a 0–1 scale. We denote all transformed measures as $y_{ij}$. Commonly
used linear transformations such as \( y_{ij} = \frac{x_{ij} - \min_{i=1}^{I} x_{ij}}{\max_{j=1}^{J} x_{ij} - \min_{i=1}^{I} x_{ij}} \) can be adopted. The score of a supplier is expressed as the weighted sum of transformed measures, \( S_i = \sum_{j=1}^{J} w_{ij} y_{ij} \), where \( w_{ij} (j = 1, 2, 3, \ldots, J) \) is the weight of criteria \( j \) of supplier \( i \).

We enable the purchasing manager to incorporate the ranking of the importance of the criteria in the decision making process. The ranking, although somehow subjective, is a far simpler requirement as compared to requirements in the AHP approach. We require the user to rank the criteria importance in a sequence, rather than specifying exact weight values or exact degrees of relative preferences. In addition, for the 23 criteria identified in research literature (Dickson, 1966; Weber, 1991), mean ratings of criteria are available, which provide a guideline for decision makers to rank the criteria. In our model development, we assume the criteria are arranged in the descending order of importance (i.e. \( w_1 \geq w_2 \geq w_3 \geq \cdots \geq w_J \)). For ease of interpretation, we assume the weights \( w_j \)'s are non-negative and are normalized so that \( \sum_{j=1}^{J} w_j = 1 \). After normalization, all scores \( S_i (i = 1, 2, 3, \ldots, I) \) are always within a 0–1 scale. The value of the weight of a particular criterion is equal to the proportion of contribution of the criterion, in the total contribution of all criteria.

We formulate our proposed multi-criteria supplier selection model as the following mathematical model (P1),

\[
\text{(P1) Max } \quad S_i = \sum_{j=1}^{J} w_{ij} y_{ij} \quad \text{(1)}
\]

s.t. \( w_{ij} - w_{i(j+1)} \geq 0 \quad j = 1, 2, 3, \ldots, (J-1), \) \quad \text{(2)}

\[
\sum_{j=1}^{J} w_{ij} = 1, \quad \text{(3)}
\]

\( w_{ij} \geq 0 \quad j = 1, 2, 3, \ldots, J. \quad \text{(4)}
\]

Constraint (2) ensures the weight values are in the same sequence as ranking. Constraint (3) is normalization. Similar to DEA, the weights are generated automatically when the maximization problem is solved and the corresponding score \( S_i \) is the maximal score supplier \( i \) can achieve.

3. Solution scheme

The proposed model (P1) is a linear optimization model. Certainly, one can solve a series of linear optimizations for all suppliers. However, this requires a linear optimizer to be available to decision makers. In addition, the processing time can be very long, especially when the number of potential suppliers is large. In this section, we adopt a transformation to simplify our model. The simplified model can be easily solved without a linear optimizer.

Denote

\[
u_{ij} = w_{ij} - w_{i(j+1)}, \quad i = 1, 2, \ldots, I \quad \text{and} \quad j = 1, 2, \ldots, (J-1)
\]

and

\[
u_{ij} = w_{ij}, \quad i = 1, 2, \ldots, I.
\]

**Lemma 1.** Constraints (2) and (3) are equivalent to the following

\[
\sum_{j=1}^{J} j \mu_{ij} = 1, \quad i = 1, 2, \ldots, I \quad \text{(7)}
\]

and

\[
u_{ij} \geq 0 \quad i = 1, 2, \ldots, I \quad \text{and} \quad j = 1, 2, \ldots, J.
\]

**Proof.** See Appendix. \( \square \)
Denote also
\[ a_{ij} = \sum_{k=1}^{j} y_{ik}, \quad i = 1, 2, \ldots, I. \] (12)

**Lemma 2.** The following relationship holds.
\[ S_i = \sum_{j=1}^{J} w_{ij}y_{ij} = \sum_{j=1}^{J} u_{ij}a_{ij}. \] (13)

**Proof.** See Appendix. □

**Lemma 3.** The non-negativity constraints for all \( w_{ij} \) are retained if all \( u_{ij} \geq 0 \)

**Proof.** See Appendix. □

**Theorem 1.** The model (P1) is equivalent to the following (P2):
\[
\text{(P2)} \quad \max \quad S_i = \sum_{j=1}^{J} u_{ij}a_{ij} \\
\text{s.t.} \quad \sum_{j=1}^{J} j u_{ij} = 1, \\
\quad u_{ij} \geq 0, j = 1, 2, \ldots, J.
\]

**Proof.** Theorem 1 is a direct result from Lemmas 1–3. □

**Theorem 2.** The optimal score \( S_i \) of the \( i \)th supplier is
\[ \max_{j=1,2,\ldots,J} \left( \frac{1}{j} \sum_{k=1}^{j} y_{ik} \right). \]

**Proof.** See Appendix. □

Based on Theorem 2, optimal score \( S_i \) can be obtained by comparing all partial averages of transformed measures. Procedure for supplier selection based on the proposed model is simple and efficient. We input all the measures of all suppliers and perform the following steps:

**Step 1.** List all measures in the same sequence as importance of criteria.
**Step 2.** Transform measures so that all measures are positively related to scores and normalized in a 0–1 scale.
**Step 3.** Calculate all partial averages, \( \frac{1}{j} \sum_{k=1}^{j} y_{ik}, \quad j = 1, 2, \ldots, J. \)
**Step 4.** Compare and locate the maximum among these partial averages. The corresponding value is the score \( S_i \) of the \( i \)th supplier.
**Step 5.** Sort the scores \( S_i \)'s in descending order and
**Step 6.** Identify important supplier(s).

The whole process requires no linear optimizer and is easy-to-implement with a common spreadsheet package which can be handled by decision makers without any background of optimization.

**4. Sensitivity analysis**

The model is simple-to-understand and easy-to-use. However, exogenous specification of ranking of criteria is required. The results may be dependent on the sequence of user-defined ranking. One can examine the sensitivity of change to a supplier score if criteria ranking changed, by studying the positioning of maximal partial averages.
Theorem 3. Only one of optimal solutions \( u_{ij} \) in (P2) is non-zero and all other are zeros and the only non-zero \( u_{ij} \) is equal to \( 1/j \).

Proof. See Appendix. \( \square \)

Theorem 4. Suppose the \( j \)th partial average is the maximal partial average. The condition \( k > j \frac{n_{k}}{u_{ij}} \) (for \( k = 1, 2, \ldots, J \) and \( j \neq k \)) on criteria ranking holds.

Proof. See Appendix. \( \square \)

The result of Theorem 4 indicates that if one finds a new ranking, causing the condition to fail, then the supplier score will change. Otherwise, the supplier score remains unchanged.

In addition to change in ranking order, any change in measure values may change the supplier score. Theorem 4 can be employed to examine the sensitivity if there is a revision of measures. If one finds that the revised value of a measure is causing the condition to fail, the supplier score will also change. This analysis is important for negotiations with suppliers. The purchasing manager can use the information to deal with a supplier to ask for any improvement needed for the supplier to be short-listed.

5. Numerical illustrations

We illustrate implementation of our proposed model with a multi-criteria supplier selection problem as in the literature [7]. Five criteria, including supply variety, quality, distance, delivery and price are under consideration by a firm manufacturing agricultural and construction equipment. Supply variety is the number of parts supplied by the supplier. It is considered first as the company would like to reduce the number of suppliers. The quality of supplied parts is also an important criterion for a company in supplier evaluation. The distance is related to delivery efficiency. A longer distance will affect the delivery service of the supplier due to a longer lead time or restricted delivery time windows. The criterion “Delivery” measures the percentage of on-time delivery. Lastly, the price index indicates the estimated price level offered by a supplier as compared to the average market price. If the price level offered is higher than the average price, the price index will be of a value higher than 100% and vice versa.

There are 18 suppliers available. The measures of each supplier under the five criteria are listed in Table 1.

Using our proposed model, we can express this supplier selection problem as follows. For each supplier \( i \), we are looking for maximal supplier score such that

\[
\text{Max} \quad \text{Supplier Score}_i = w_{i1}(\text{Supply Variety}) + w_{i2}(\text{Quality}) + w_{i3}(\text{Reciprocal of Distance}) + w_{i4}(\text{Delivery}) + w_{i5}(\text{Reciprocal of Price Index})
\]

\[
\text{s.t.} \quad w_{i1} - w_{i2} \geq 0,
\]

\[
w_{i2} - w_{i3} \geq 0,
\]

\[
w_{i3} - w_{i4} \geq 0,
\]

\[
w_{i4} - w_{i5} \geq 0,
\]

\[
w_{i1} + w_{i2} + w_{i3} + w_{i4} + w_{i5} = 1,
\]

\[
w_{i1}, w_{i2}, w_{i3}, w_{i4}, w_{i5} \geq 0.
\]

The above linear model can be solved by comparing the partial averages rather than using the optimization procedure. Partial averages are calculated and the maximal scores are located by choosing the largest partial averages. Table 3 shows our results.

For comparison purpose, we consider the best 5 suppliers as there were 5 efficient suppliers identified by the DEA model in [6], using the same dataset (refer to Appendix). The top 5 suppliers identified are suppliers 10,
Suppliers 10, 15 and 17 are good suppliers in both DEA and the proposed model. Suppliers 5 and 11 were not identified as good suppliers in the DEA model. On the other hand, suppliers 1 and 12 were identified as good suppliers in the DEA model but were not identified by our proposed model. The reason for these differences are due to the incorporation of the relative importance of the criteria. Suppliers 1 and 12 were efficient suppliers in DEA models. However, the supply varieties of these two suppliers are only 2 and 7, which are relatively low, compared to other suppliers. When the supply variety is considered as a relatively important criterion, these two suppliers are eliminated. The good suppliers we identified are good not simply by the most important criterion (supply variety). Suppliers 5 and 11 with relatively low supply variety measures, 24 and 10 respectively, were rated high because of the advantage of relatively shorter distances.

### Table 1
Measures of suppliers under criteria

<table>
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<tr>
<th>Supplier number</th>
<th>Supply variety (Unit)</th>
<th>Quality (%)</th>
<th>Distance (Mile)</th>
<th>Delivery (%)</th>
<th>Price index (%)</th>
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<td>97</td>
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### Table 2
Transformed and normalized measures of suppliers under criteria

<table>
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<tr>
<th>Supplier number</th>
<th>Normalized measures</th>
<th>Supply variety</th>
<th>Quality</th>
<th>Reciprocal of distance</th>
<th>Delivery</th>
<th>Reciprocal of price index</th>
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In addition to comparison of the results against those from a DEA model, we can examine the changes that may have been caused by the order of criteria ranking. Suppose the company would like to swap the order of importance of criteria delivery and price. We can apply the condition in Theorem 4. There are only two suppliers (suppliers 11 and 12) who failed to hold their positions under the new ranking. The new supplier scores for these two suppliers are re-calculated as 0.718 for supplier 11 and 0.548 for supplier 12. Scores of other suppliers remain unchanged as the condition in Theorem 4 holds. Under the revised criteria ranking, supplier 11 is no longer short-listed as one of the top 5 suppliers. Supplier 8 with a score of 0.72 becomes one of the best suppliers.

Re-considering the problem with the original criteria ranking order, we would like to know improvements the 6th supplier (supplier 8) needs so as to be short-listed among the top 5 good suppliers. Supplier 8 currently has a score of 0.72 and the normalized transformed measures of the criteria supply variety, quality, reciprocal of distance, delivery and reciprocal of price are 0.44, 1.00, 0.13, 0.85 and 0.00, respectively. Changes in these values, causing the condition in Theorem 4 to fail, are as follows:

1. Improve supply variation by more than 0.01 in the normalized scale or increase by 1 in the original scale.
2. Improve reciprocal of distance by more than 0.48 or lessen by distance by 620 miles in the original scale.
3. Improve delivery by more than 0.47 in the normalized scale which is infeasible.
4. Improve reciprocal of price by more than 1.18 in the normalized scale which is infeasible.

According to the above results, the purchasing manager can negotiate with supplier 8 to see if the supplier can increase supply variation by 1 or shorten the distance by 620 miles.

### 6. Summary and conclusions

In this paper, we propose a weighted linear program for the multi-criteria supplier selection problem. The proposed model retains the advantages of the DEA non-parametric approach. However, it allows involvement of the decision maker in ranking the relativity of importance of criteria. In addition, the decision maker’s role is not as subjective as other approaches in AHP or MOP models. Moreover, we present a transformation technique which enables the weighted linear program to be solved without optimization. This model for
the multi-criteria supplier selection problem can be easily implemented with a spreadsheet package. The model can be widely applied to practical situations and does not require users with any optimization background.

Appendix

Proof of Lemma 1. Eq. (8) is a direct result of constraint (2). We also need to prove the equivalence between (3) and (7). Substituting (5) into left-hand-side of (7), we have

$$\sum_{j=1}^{J} j u_{ij} = u_{i1} + 2u_{i2} + 3u_{i3} + \cdots + J u_{ij}$$

(9)

$$= (w_{i1} - w_{i2}) + 2(w_{i2} - w_{i3}) + 3(w_{i3} - w_{i4}) + \cdots + J(w_{in})$$

(10)

$$= \sum_{j=1}^{J} w_{ij}.$$ 

(11)

This completes the proof. □

Proof of Lemma 2. Substituting (5) and (12) into the right-hand-side of (13), we have

$$\sum_{j=1}^{J} u_{ij} a_{ij} = \sum_{j=1}^{J-1} \left( \left( w_{ij} - w_{i(j+1)} \right) \sum_{k=1}^{J} y_{ik} \right) + u_{ij} \sum_{k=1}^{J} y_{ik}$$

$$= \left( (w_{i1} - w_{i2})y_{i1} + (w_{i2} - w_{i3})y_{i2} + (w_{i3} - w_{i4})y_{i3} + \cdots \right) + \left( (w_{i(J-1)} - w_{ij})y_{i1} + \cdots + y_{i(j-1)} \right) + u_{ij} (y_{i1} + y_{i2} + \cdots + y_{ij})$$

$$= w_{i1}y_{i1} + w_{i2}y_{i2} + \cdots + w_{ij}y_{ij}$$

$$= \sum_{j=1}^{J} w_{ij} y_{ij}.$$ 

The proof is completed. □

Proof of Lemma 3. From (5) and (6), we can have the following for all item $i$.

$$w_{ij} = \sum_{k=j}^{J} u_{ik}; j = 1, 2, \ldots, J.$$ 

(14)

If $u_{ij} \geq 0 \ (j = 1, 2, \ldots, J)$, we retain the non-negativity constraint for all $w_{ij}$.

This completes the proof. □

Proof of Theorem 2. The dual formulation of (P2) is as in the following, with a dual variable $z_i$.

(P2 dual) \[ \min z_i \]

s.t. $z_i \geq \frac{1}{j} a_{ij}, \quad j = 1, 2, \ldots, J.$

The minimal $z_i$ for item $i$ can be directly obtained as $\max_{j=1, 2, \ldots, J} \left( \frac{1}{j} a_{ij} \right)$, which is $\max_{j=1, 2, \ldots, J} \left( \frac{1}{J} \sum_{k=1}^{J} y_{ik} \right)$ in terms of decision variables in original problem (P1). □

Proof of Theorem 3. Considering the dual formulation (P2 dual), there is only one binding constraint when (P2 dual) is optimized. Therefore, there is only one non-zero optimal solution $u_{ij}$ in (P2). Owing to Eq. (7), the only non-zero $u_{ij}$ is equal to $1/j$. □
Proof of Theorem 4. Consider the dual formulation (P2 dual), for $j$th partial average to be the maximal, we have $\frac{1}{j}x_{ij} > \frac{1}{k}x_{ik}$ (for $k = 1, 2, \ldots, J$ and $j \neq k$). Rearranging the terms, we have the condition $k > j \frac{x_{ik}}{x_{ij}}$, for $k = 1, 2, \ldots, J$ and $j \neq k$. □

References