An Interactive Approach for Multiple Criteria Selection Problem

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   - Interactive Algorithms

3 Interactive Algorithm
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   - Likelihood Estimation
   - Finding the Alternatives Eliminated by Each Cone
   - Steps of the Algorithm

4 An Illustrative Example

5 Conclusion
Introduction

- Most problems in nature have several objectives to be satisfied
- Multiple criteria decision making (MCDM): decision making with multiple and conflicting criteria
- Decision makers preferences affect directly the solution
- MCDM problems are widespread in our life
Two classes of problems in MCDM:

- Evaluation problem: A finite number of alternatives are explicitly known
- Design problem: Alternatives are not explicitly known, can be identified by solving mathematical programming models.
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Four sub-classes of evaluation problems:

- Choice problem
- Ranking problem
- Sorting problem
- Classification problem
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Four sub-classes of **evaluation problems**:

- **Choice problem**
- Ranking problem
- Sorting problem
- Classification problem
Introduction

- Selecting the best alternative among several alternatives
- Each alternative is evaluated on multiple criteria
- Finding the best solution with the minimum number of questions
- We develop an interactive algorithm
Introduction

- $m$ alternatives, $p$ criteria and one decision maker (DM)
- We assume DM has a quasiconcave value function which is not explicitly known
- Quasiconcave value functions represent human nature properly
- We gather information about the value function through pairwise comparisons provided by DM
- Inferior alternatives are identified and eliminated according to DM’s responses
Theorem

(Korhonen, Wallenius and Zionts, 1984) Assume a quasi-concave and nondecreasing function \( f(x) \) defined in a \( p \)-dimensional Euclidean space \( \mathbb{R}^p \). Consider distinct points \( x_i \in \mathbb{R}^p, i = 1, ..., m \), and any point \( x^* \in \mathbb{R}^p \) and assume that \( f(x_k) < f(x_i), i \neq k \). Then, if \( \epsilon \geq 0 \) in the following linear programming problem

\[
\begin{align*}
\text{Max} & \quad \epsilon \\
\text{s.t.} & \quad \sum_{i=1}^{m} \mu_i (x_k - x_i) - \epsilon \geq x^* - x_k \\
& \quad \mu_i \geq 0, \forall i
\end{align*}
\]

It follows that \( f(x_k) \geq f(x^*) \).
- Single cone:
  If $X_i$ is preferred to $X_j$
  $Cone(X_i, X_j)$

- Double cone:
  If $X_i$ is preferred to $X_k$, and $X_j$ is preferred to $X_k$
  $Cone(X_i, X_j, X_k)$

- In general; $m$ – cone:
  If $m$ alternatives are preferred to $(m + 1)^{st}$ alternative
- $X_1$, $X_2$, $X_3$ and $X_4$ are non-dominated alternatives.
- DM prefers $X_1$ to $X_2$, then all alternatives in Region A are dominated by $(X_1, X_2)$ cone.
- $X_2$ and $X_3$ cannot be the best alternative.
- Continue with pairwise comparisons until only one alternative is left, which is the best alternative.

Figure: The Illustration of Cone
Köksalan, Karwan and Zionts (1984) use dummy alternatives in order to reduce the number of questions.
Köksalan and Taner (1989) develop variations of the dummy alternatives.
Convex Cone Approach

- Köksalan (1989) uses the ideal point as an evaluation criterion to reduce the number of questions.
- Taner and Köksalan (1991) conduct experiments to see the effect of cones.
- Karsu (2013) reviews the theory of convex cones approach.
Interactive Algorithms

Gather information from the DM when needed throughout the algorithm.

- Buğdacı et al. (2013) propose an interactive probabilistic sorting method.
Malakooti (1988) and Malakooti (1989) propose heuristic and exact algorithms to identify and eliminate inefficient alternatives, hence reducing the pairwise comparisons required.

Karsu et al. (2012) propose an interactive ranking method using convex cones.
An approach to compute the likelihood that DM will prefer an alternative to another

Estimate the maximum and minimum value of each alternative $X_i$

$f_{max}(i)$ and $f_{min}(i)$, respectively.
Estimating the Value Ranges

**Maximum(Minimum) Value Model:**

\[
\begin{align*}
\text{Max(Min)} \ Z &= \mu_i \\
\text{s.t.} \\
\mu_i &= \sum_{q=1}^{p} x_{iq} w_q, \forall i \\
\mu_i &\geq \mu_k, X_i \succ X_k \\
\sum_{q=1}^{p} w_q &= 1 \\
\mu_i &\geq 0, \forall i \\
w_q &\geq 0, \forall q
\end{align*}
\]
Likelihood Estimation

- $P(i, k)$: likelihood that alternative $X_i$ is preferred to alternative $X_k$
- Uniform probability distribution
Likelihood Estimation

For alternatives $X_i$ and $X_k$ such that $f_{max}(i) \geq f_{max}(k)$:

- **Case 1:**
  
  $f_{max}(i) \geq f_{max}(k) \geq f_{min}(i) \geq f_{min}(k)$

  
  $P(i, k) = \frac{f_{max}(i) - \left(\frac{f_{max}(k) + f_{min}(i)}{2}\right) + f_{min}(i) - f_{min}(k)}{f_{max}(i) - f_{min}(k)}$

- **Case 2:** $f_{max}(i) \geq f_{max}(k)$ and $f_{min}(i) \leq f_{min}(k)$

  
  $P(i, k) = \frac{f_{max}(i) - \left(\frac{f_{max}(k) + f_{min}(k)}{2}\right)}{f_{max}(i) - f_{min}(i)}$

- **Case 3:** $f_{min}(i) \geq f_{max}(k)$

  
  $P(i, k) = 1$

  and $P(k, i) = 1 - P(i, k)$
Finding the Alternatives Eliminated by Each Cone

- If DM prefers $X_i$ to $X_k$ $\Rightarrow$ $Cone(X_i, X_k)$
- Does $Cone(X_i, X_k)$ dominate alternative $X_t$ ?
- A model that considers all $(X_i, X_k, X_t)$ triplets
Mathematical Model

**Combined Cone Model:**

\[
\begin{align*}
\text{Max } Z &= \sum_{i=1}^{m} \sum_{k=1}^{m} \sum_{t=1}^{m} \epsilon_{ikt} \\
\text{s.t. } \\
\mu_{ikt}(x_{kq} - x_{iq}) - \epsilon_{ikt} &\geq x_{tq} - x_{kq}, \forall q, i, k, t \\
\mu_{ikt} &\geq 0, \forall i, k, t
\end{align*}
\]
Mathematical Model

- Let $\epsilon^{*}_{ikt}$ be the value at the optimal solution
- If $\epsilon^{*}_{ikt} \geq 0 \Rightarrow \text{Cone}(X_i, X_k)$ dominates $X_t$
- $NE(i, k)$: the number of alternatives $\text{Cone}(X_i, X_k)$ dominates:

$$NE(i, k) = \sum_{t=1, t \neq i, t \neq k}^{m} 1$$

- $E[i, k]$: Expected number of alternatives to be eliminated by $(X_i, X_k)$ comparison

$$E[i, k] = P(i, k) \times NE(i, k) + P(k, i) \times NE(k, i)$$
Interactive Algorithm

0 Eliminate the dominated alternatives.
Interactive Algorithm

0. Eliminate the dominated alternatives.

1. Solve combined cone model and compute initial $NE(i, k)$ values.
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2. Find $f_{min}(i)$ and $f_{max}(i)$ scores (in case of infeasibility, remove the oldest constraints one by one until feasibility is reached).
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3. Compute $P(i, k)$ and $E[i, k]$.
4. Pick the alternative pair $(X_i, X_k)$ with the highest $E[i, k]$ value and ask DM to compare these alternatives. Assume that DM prefers alternative $X_i$ to alternative $X_k$.
5. Eliminate alternatives dominated by $Cone(X_i, X_k)$, add a new constraint to the model and update $NE(i, k)$ values.
6. If there is only one alternative left, go to Step 7 otherwise go to Step 2.
7. Report the remaining alternative as the most preferred one and stop.
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Interactive Algorithm

Computational effort

- **Number of iterations**
  - Initially, there are $m$ alternatives
  - At least, one alternative is eliminated in each iteration
  - The algorithm terminates in at most $m - 1$ iterations
Interactive Algorithm

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  - Initially, there are $m$ alternatives
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- **At each iteration**
  - Two LP problems for each remaining alternative $X_i$
  - $NE(i, k)$ and $E[i, k]$ computations for each pair of remaining alternatives $X_i$ and $X_k$
Illustrative Example

- \( m = 9 \) alternatives evaluated on \( p = 3 \) criteria \((u_1, u_2, u_3)\)
- DM has an underlying quadratic value function as below:
  \[-(u_1 - 66)^2 - (u_2 - 80)^2 - (u_3 - 75)^2\]
- Used to simulate DM’s responses to pairwise comparisons

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>66</td>
<td>30</td>
<td>-12</td>
<td>-10069</td>
</tr>
<tr>
<td>2</td>
<td>48</td>
<td>60</td>
<td>12</td>
<td>-4693</td>
</tr>
<tr>
<td>3</td>
<td>36</td>
<td>12</td>
<td>72</td>
<td>-5533</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>66</td>
<td>66</td>
<td>-2041</td>
</tr>
<tr>
<td>5</td>
<td>60</td>
<td>20</td>
<td>-20</td>
<td>-12661</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td>-15</td>
<td>75</td>
<td>-11626</td>
</tr>
<tr>
<td>7</td>
<td>30</td>
<td>30</td>
<td>15</td>
<td>-7396</td>
</tr>
<tr>
<td>8</td>
<td>20</td>
<td>80</td>
<td>40</td>
<td>-3341</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-16381</td>
</tr>
</tbody>
</table>
Illustrative Example

- Step 0. Eliminate dominated alternatives $X_5$ and $X_9$
- Step 1. Obtain the number of eliminated alternatives by each cone
- Step 2. Compute $f_{min}(i)$ and $f_{max}(i)$ values for each $X_i$

**Table:** Estimated value intervals of the alternatives (Initial)

<table>
<thead>
<tr>
<th>Alternative($i$)</th>
<th>$f_{min}(i)$</th>
<th>$f_{max}(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>66</td>
</tr>
<tr>
<td>2</td>
<td>17.5</td>
<td>54</td>
</tr>
<tr>
<td>3</td>
<td>22</td>
<td>66.5</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>66</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>65.8</td>
</tr>
<tr>
<td>7</td>
<td>17.3</td>
<td>30</td>
</tr>
<tr>
<td>8</td>
<td>20</td>
<td>73.3</td>
</tr>
</tbody>
</table>
Illustrative Example

Step 3. Compute $P(i, k)$ and $E[i, k]$

Table: Sample computation of $P(i, k)$ values (Intermediate Iteration)

<table>
<thead>
<tr>
<th>$i$</th>
<th>$f_{\text{min}}(i)$</th>
<th>$f_{\text{max}}(i)$</th>
<th>$k$</th>
<th>$f_{\text{min}}(k)$</th>
<th>$f_{\text{max}}(k)$</th>
<th>Case</th>
<th>$P(i, k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>31.2</td>
<td>66.5</td>
<td>1</td>
<td>0.0</td>
<td>49.1</td>
<td>1</td>
<td>0.865</td>
</tr>
<tr>
<td>1</td>
<td>0.0</td>
<td>49.1</td>
<td>2</td>
<td>17.5</td>
<td>49.1</td>
<td>2</td>
<td>0.321</td>
</tr>
<tr>
<td>4</td>
<td>40.1</td>
<td>66.0</td>
<td>7</td>
<td>17.3</td>
<td>28.9</td>
<td>3</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table: Sample computation of $E[i, k]$ values (Intermediate Iteration)

<table>
<thead>
<tr>
<th>$(i, k)$</th>
<th>$NE(i, k)$</th>
<th>$NE(k, i)$</th>
<th>$P(i, k)$</th>
<th>$P(k, i)$</th>
<th>$E[i, k]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3,1)</td>
<td>2</td>
<td>1</td>
<td>0.865</td>
<td>0.135</td>
<td>1.865</td>
</tr>
<tr>
<td>(1,2)</td>
<td>1</td>
<td>3</td>
<td>0.321</td>
<td>0.679</td>
<td>2.357</td>
</tr>
<tr>
<td>(4,7)</td>
<td>3</td>
<td>1</td>
<td>1.000</td>
<td>0.000</td>
<td>3.000</td>
</tr>
</tbody>
</table>
Step 4. Ask DM to compare the alternative pair \((X_i, X_k)\) with the highest \(E[i, k]\) value
Assume DM prefers \(X_i\) to \(X_k\)

Step 5. Eliminate the alternatives \(Cone(X_i, X_k)\) dominates

Step 6. Go to Step 2 until only one alternative is remaining

Step 7. Report the remaining alternative as the most preferred one
Conclusions

- Multiple criteria selection problem
- Convex cones
- An interactive algorithm
Future work

- Computational tests
- Variations of the algorithm
- Comparisons with other approaches
Questions and comments

Thanks for your attention

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