Decision Support

Building a set of additive value functions representing a reference preorder and intensities of preference: GRIP method

José Rui Figueira a,1,2, Salvatore Greco b,3, Roman Słowiński c,d,*

a CEG-IST, Center for Management Studies, Instituto Superior Técnico, Technical University of Lisbon, Tagus Park, Av. Cavaco Silva, 2780-990 Porto Salvo, Portugal
b Faculty of Economics, University of Catania, Corso Italia 55, 95 129 Catania, Italy
c Institute of Computing Science, Poznań University of Technology, Street Piotrowo 2, 60-965 Poznań, Poland
d Institute for Systems Research, Polish Academy of Sciences, 01-447 Warsaw, Poland

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Abstract

We present a method called Generalized Regression with Intensities of Preference (GRIP) for ranking a finite set of actions evaluated on multiple criteria. GRIP builds a set of additive value functions compatible with preference information composed of a partial preorder and required intensities of preference on a subset of actions, called reference actions. It constructs not only the preference relation in the considered set of actions, but it also gives information about intensities of preference for pairs of actions from this set for a given decision maker (DM). Distinguishing necessary and possible consequences of preference information on the considered set of actions, GRIP answers questions of robustness analysis. The proposed methodology can be seen as an extension of the UTA method based on ordinal regression. GRIP can also be compared to the AHP method, which requires pairwise comparison of all actions and criteria, and yields a priority ranking of actions. As for the preference information being used, GRIP can be compared, moreover, to the MACBETH method which also takes into account a preference order of actions and intensity of preference for pairs of actions. The preference information used in GRIP does not need, however, to be complete: the DM is asked to provide comparisons of only those pairs of reference actions on particular criteria for which his/her judgment is sufficiently certain. This is an important advantage comparing to methods which, instead, require comparison of all possible pairs of actions on all the considered criteria. Moreover, GRIP works with a set of general additive value functions compatible with the preference information, while other methods use a single and less general value function, such as the weighted-sum.

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1. Introduction

Ranking a finite set of actions evaluated on a finite set of criteria is a problem of uttermost importance in many real-world areas of decision-making (see Figueira et al., 2005). Among many approaches that have been designed to deal with the ranking problem, two of them seem to prevail. The first one exploits the idea of assigning a score to each action, as it is
the case of multi-attribute utility theory (MAUT) (see Keeney and Raiffa, 1976). The second relies on the principle of pairwise comparison of actions, as it is the case of outranking methods (see Roy, 1996). The value function and the outranking relation are two preference models underlying these two main approaches. In order to build such models, preference information from the Decision Maker (DM) is required.

The preference information may be either direct or indirect, depending whether it specifies directly values of some parameters used in the preference model (e.g. tradeoff weights, aspiration levels, discrimination thresholds, etc.) or whether it specifies some examples of holistic judgments from which compatible values of the preference model parameters are induced. Eliciting direct preference information from the DM can be counterproductive in real-world decision-making because of a high cognitive effort required. Consequently, asking directly the DM to provide values for the parameters seems to make the DM uncomfortable. Eliciting indirect preference is less demanding of the cognitive effort. Indirect preference information is mainly used in the ordinal regression paradigm. According to this paradigm, a holistic preference information on a subset of some reference or training actions is known first and then a preference model compatible with the information is built and applied to the whole set of actions in order to rank them.

The ordinal regression paradigm is concordant with the posterior rationality postulated by March (1978). It has been known for at least fifty years in the field of multidimensional analysis. It is also concordant with the induction principle used in machine learning. This paradigm has been applied within the two main multiple criteria decision analysis (MCDA) approaches mentioned above: the one using a value function as a preference model (Srinivasan and Shocker, 1973; Pekelman and Sen, 1974; Jacquet-Lagrère and Siskos, 1982; Siskos et al., 2005), and the one using an outranking relation as a preference model (Kiss et al., 1994; Mousseau and Słowiński, 1998). This paradigm has also been used since mid ninetieth’s in MCDA methods involving a new, third family of preference models – a set of dominance decision rules induced from rough approximations of holistic preference relations (Greco et al. 1999, 2001, 2003, 2005; Słowiński et al., 2005).

Recently, the ordinal regression paradigm has been revisited with the aim of considering the whole set of value functions compatible with the preference information provided by the DM, instead of a single compatible value function used, for example, in the UTA-like methods (Jacquet-Lagrère and Siskos, 1982; Siskos et al., 2005). This extension has been implemented in a method called UTA GMS (Greco et al., 2003, in press). This method is not revealing to the DM one compatible value function, but it is using the whole set of compatible (general, not piecewise linear only) additive value functions to set up a necessary weak preference relation and a possible weak preference relation in the whole set of considered actions. Moreover, in UTA GMS, the preference information has the form of a partial preorder (instead of a complete preorder) in a subset of reference actions. The necessary and possible weak preference relations are exploited such that one finally obtains two rankings in the set of actions: the necessary ranking (partial preorder) identifying preference statements being true for all compatible value functions, and the possible ranking (complete and negatively transitive binary relation) identifying preference statements being true for at least one compatible value function. Distinguishing necessary and possible consequences of preference information on the considered set of actions, UTA GMS answers questions of robustness analysis (Roy, 1998).

In this paper, we present a new method called GRIP (Generalized Regression with Intensities of Preference) which also belongs to the class of methods based on indirect preference information and the ordinal regression paradigm. GRIP generalizes both the UTA and UTA GMS methods by adopting all features of UTA GMS and taking into account additional preference information in the form of comparisons of intensities of preference between some pairs of reference actions. These comparisons are expressed in two possible ways (not exclusive): comprehensively, i.e. on all criteria, and partially, i.e. on particular criteria.

GRIP can be compared to the AHP method (Saaty, 2005), which requires, from the DM, preference information composed of pairwise comparisons of all actions and criteria on a fixed ratio scale, and constructs a weighted-sum value function producing a priority ranking of actions.

From the viewpoint of the type of preference information being used, the GRIP method can be compared, moreover, to the MACBETH method (Bana e Costa and Vansnick, 1994; Bana e Costa et al., 2005), which also involves a preference order in the set of actions and the intensity of preference for pairs of actions. Unlike in MACBETH, however, the preference information in GRIP does not need to be complete, i.e. the preference order may be partial and may concern just reference actions, not all actions, and, moreover, information about intensities of preference may also concern some pairs of reference actions, not all possible pairs of actions. This is an important feature of GRIP, answering to the current demand commonly addressed to decision aiding methods: “try to help DMs using incomplete although reliable information”.

The paper is organized in the following way. In Section 2, we present motivations that led us to build up a new method. In Section 3, we recall some useful concepts concerning mathematical modeling of preferences, with an adequate notation. Sections 4 and 5 are devoted to the UTA and UTA GMS methods, respectively. In Section 6, we present the new GRIP method and we provide a theoretical comparison of GRIP with AHP and MACBETH. Finally, Section 7 provides conclusions and avenues for future research.
2. Motivations

Apart from the reasons that motivated the proposal of the UTA\textsuperscript{GMS} method (Greco et al., in press), there are two major issues that led us to conduct research on this very topic:

- **Preference information.** There is a need, observed in practice, of handling information related to intensity of preferences that was not considered in the UTA-like methods. In many real-life decision-making situations, DMs are willing to provide more information than a partial preorder on a set of reference or training actions. Thus, it is frequent to observe assertions of the type “x is preferred to y at least as much as w is preferred to z”, expressed on particular criteria (partially) and/or on all criteria together (comprehensively).

- **Technical aspects.** The additional constraints related to requirements about intensity of preference can reduce the feasible polyhedron of all value functions compatible with preference information – the polyhedron which, in general, can be quite large. This can be useful for both the classical UTA method and the UTA\textsuperscript{GMS} method.

Similarly to UTA\textsuperscript{GMS}, it would be desirable to design an interactive procedure based on the GRIP methodology, enabling progressive articulation of DM’s preferences along with narrowing the range of compatible value functions, in the spirit of a constructive (or learning) process.

3. Mathematical background on preference modeling

This section aims to recall some basic concepts of MCDA and mathematical preference modeling, along with an adequate notation.

3.1. Elementary notation and problem statement

We are considering a multiple criteria decision problem where a finite set of actions \( A = \{x, \ldots, y, \ldots, w, \ldots, z\} \) is evaluated on a family \( F = \{g_1, g_2, \ldots, g_n\} \) of \( n \) criteria. Let \( I = \{1, 2, \ldots, n\} \) denote the set of criteria indices. We assume, without loss of generality, that the greater \( g_i(x) \), the better action \( x \) on criterion \( g_i \), for all \( i \in I \), \( x \in A \). A DM is willing to rank the actions of \( A \) from the best to the worst, according to his/her preferences. The ranking can be complete or partial, depending on the preference information provided by the DM and on the way of exploiting this information. The family of criteria \( F \) is supposed to satisfy consistency conditions, i.e. completeness (all relevant criteria are considered), monotonicity (the better the evaluation of an action on considered criteria, the more it is preferable to another), and non-redundancy (no superfluous criteria are considered) (see Roy and Bouyssou, 1993).

Such a decision-making problem statement is called *multiple criteria ranking problem*. It is known that the only objective information coming out from the formulation of this problem is the dominance ranking. Let us recall that in the dominance ranking, action \( x \in A \) is preferred to action \( y \in A \) if and only if \( g_i(x) \geq g_i(y) \) for all \( i \in I \), with at least one strict inequality. Moreover, \( x \) is indifferent to \( y \) if and only if \( g_i(x) = g_i(y) \) for all \( i \in I \). Hence, for any two actions \( x, y \in A \), one of the four situations may arise in the dominance ranking: \( x \) is preferred to \( y \), \( y \) is preferred to \( x \), \( x \) is indifferent to \( y \), or \( x \) is incomparable to \( y \). Usually, the dominance ranking is very poor, i.e. the most frequent situation is that \( x \) is incomparable to \( y \).

In order to enrich the dominance ranking, the DM has to provide preference information which is used to construct an aggregation model making the actions more comparable. Such an aggregation model is called preference model. It induces a preference structure on set \( A \), whose proper exploitation permits to work out a ranking proposed to the DM.

In what follows, the evaluation of each action \( x \in A \) on each criterion \( g_i \in F \) will be denoted either by \( g_i(x) \) or \( x_i \). Let \( G_i \) denote the value set (scale) of criterion \( g_i \), \( i \in I \). Consequently,

\[
G = \prod_{i \in I} G_i
\]

represents the evaluation space, and \( x \in G \) denotes a profile of an action in such a space.

From a pragmatic point of view, it is reasonable to assume that \( G_i \in \mathbb{R} \) for \( i = 1, \ldots, n \). More specifically, we will assume that the evaluation scale on each criterion \( g_i \) is bounded, such that \( G_i = [x_i, \beta_i] \), \( x_i < \beta_i \), where \( x_i \), \( \beta_i \) are the worst and the best (finite) evaluations, respectively. Thus, \( g_i : A \rightarrow G_i \), \( i \in I \), therefore, each action \( x \in A \) is associated with an evaluation vector denoted by \( g(x) = (x_1, x_2, \ldots, x_n) \in G \).

4. Ordinal regression: The foundations of the UTA method

This section presents an outline of the principle of the ordinal regression via linear programming, as proposed in the original UTA method (see Jacquet-Lagrèze and Siskos, 1982).
4.1. Preference information

The preference information is given in the form of a complete preorder $\succeq$ on a subset of reference actions $A^R \subseteq A$ (where $|A^R| = p$), called reference preorder. The reference actions are usually those contained in set $A$ for which the DM is able to express holistic preferences. In addition to the actions to be ranked by the method, set $A$ can also contain some fictitious, past or other auxiliary actions. Let $A^R = \{a, b, c, \ldots\}$ be the set of reference actions.

We consider a weak preference relation $\succ$ on $A^R$ which means, for each pair of vectors, $x, y \in G$,

$$x \succ y \iff x \text{ is at least as good as } y.$$  

This weak preference relation can be decomposed into its asymmetric and symmetric parts, as follows,

1. $x \succ y \equiv [x \succeq y \text{ and not } (y \succeq x)] \iff "x \text{ is preferred to } y"$, and
2. $x \sim y \equiv [x \succeq y \text{ and } y \succeq x] \iff "x \text{ is indifferent to } y"$.

4.2. An additive model

The additive value function is defined on $A$ such that for each $g(x) \in G$,

$$U(g(x)) = \sum_{i \in I} u_i(g_i(x)),$$

where $u_i$ are non-decreasing marginal value functions, $u_i : G_i \rightarrow \mathbb{R}$, $i \in I$. For the sake of simplicity, we shall write (1) as follows,

$$U(x) = \sum_{i \in I} u_i(x_i).$$ (1')

In the UTA method, the marginal value functions $u_i$ are assumed to be piecewise linear functions. The ranges $[x_i, \beta_i]$ are divided into $\gamma_i \geq 1$ equal sub-intervals,

$$[x_i^0, x_i^1], [x_i^1, x_i^2], \ldots, [x_i^{\gamma_i - 1}, x_i^{\gamma_i}],$$

where

$$x_i^j = x_i + \frac{j}{\gamma_i}(\beta_i - x_i), \quad j = 0, \ldots, \gamma_i, \quad \text{ and } i \in I.$$

The marginal value of an action $x \in A$ is obtained by linear interpolation,

$$u_i(x_i) = u_i(x_i^j) + \frac{x_i - x_i^j}{x_i^{j+1} - x_i^j}(u_i(x_i^{j+1}) - u_i(x_i^j)), \quad \text{ for } x_i \in [x_i^j, x_i^{j+1}].$$ (2)

The piecewise linear additive model is completely defined by the marginal values at the breakpoints, i.e.

$$u_i(x_i^0) = u_i(x_i^1), u_i(x_i^2), \ldots, u_i(x_i^{\gamma_i}) = u_i(\beta_i).$$

In what follows, the principle of the UTA method is described as it was recently presented by Siskos et al. (2005).

Therefore, a value function $U(x) = \sum_{i=1}^n u_i(x_i)$ is compatible if it satisfies the following set of constraints

$$U(a) > U(b) \iff a \succ b$$

$$U(a) = U(b) \iff a \sim b$$

$$u_i(x_i^{j+1}) - u_i(x_i^j) \geq 0, \quad i = 1, \ldots, n, \quad j = 0, \ldots, \gamma_i - 1,$$

$$u_i(x_i) = 0, \quad i = 1, \ldots, n,$$

$$\sum_{i=1}^n u_i(\beta_i) = 1.$$ (3)

4.3. Checking for compatible value functions through linear programming

To verify if a compatible value function $U(x) = \sum_{i=1}^n u_i(x_i)$ restoring the reference preorder $\succ$ on $A^R$ exists, one can solve the following linear programming problem, where $u_i(x_i^j), i = 1, \ldots, n, j = 1, \ldots, \gamma_i$, are unknown, and $\sigma^+(a), \sigma^-(a), a \in A^R$, are auxiliary variables:
Min $F = \sum_{a \in A} (\sigma^+(a) + \sigma^-(a))$

subject to:

\begin{align*}
U(a) + \sigma^+(a) - \sigma^-(a) & \geq \forall a, b \in A^R, \\
U(b) + \sigma^+(b) - \sigma^-(b) & \rightleftharpoons a \succ b
\end{align*}

(4)

where $\varepsilon$ is an arbitrarily small positive value so that $U(a) + \sigma^+(a) - \sigma^-(a) \geq U(b) + \sigma^+(b) - \sigma^-(b) \in case of $a \succ b$.

If the optimal value of the objective function of program (4) is equal to zero ($F^* = 0$), then there exists at least one value function $U(x) = \sum_{i=1}^{n} u_i(x_i)$ satisfying (3), i.e. compatible with the reference preorder on $A^R$. In other words, this means that the corresponding polyhedron (3) of feasible solutions for $u_i(x_i), i = 1, \ldots, n, j = 0, \ldots, \gamma_i - 1$, $u_i(x_i) = 0, i = 1, \ldots, n$,

\[
\sum_{i=1}^{n} u_i(\beta_i) = 1,
\]

\[
\sigma^+(a), \sigma^-(a) \geq 0, \forall a \in A^R,
\]

Let us remark that the transition from the preorder $\succ$ to the marginal value function exploits the ordinal character of the criterion scale $G_i$. Note, however, that the scale of the marginal value function is a conjoint interval scale (see e.g. Theorem 13 in Chapter 6 of Krantz et al., 1971, or Theorem III.4.1 in Wakker, 1989). More precisely, for the considered additive value function, the admissible transformations of the marginal value functions $u_i(x_i)$ have the form $u'_i(x_i) = k \times u_i(x_i) + h_i, h_i \in \mathbb{R}, i = 1, \ldots, n, k > 0$, such that for all $[x_1, \ldots, x_n], [y_1, \ldots, y_n] \in G$,

\[
\sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i) \iff \sum_{i=1}^{n} u'_i(x_i) \geq \sum_{i=1}^{n} u'_i(y_i).
\]

An alternative way of representing the same preference model is:

\[
U(x) = \sum_{i \in I} w_i u_i(x) \quad \text{where } u_i(x_i) = 0, \; \hat{u}(\beta_i) = 1, \; w_i \geq 0 \; \forall i \in I, \; \text{and} \; \sum_{i \in I} w_i = 1.
\]

Note that the correspondence between (5) and (1) is such that $w_i = u_i(\beta_i), \forall i \in I$. Due to the cardinal character of the marginal value function scale, the parameters $w_i$ can be interpreted as tradeoff weights among marginal value functions $u_i(x_i)$.

We will use, however, the preference model (1) with normalization constraints bounding $U(x)$ to the interval $[0, 1]$.

When the optimal value of the objective function of the program (4) is greater than zero ($F^* > 0$), then there is no value function $U(x) = \sum_{i=1}^{n} u_i(x_i)$ compatible with the reference preorder on $A^R$. In such a case, three possible moves can be considered:

- increasing the number of linear pieces $\gamma_i$ for one or several marginal value functions $u_i$ could make it possible to find an additive value function compatible with the reference preorder on $A^R$,
- revising the reference preorder on $A^R$ could lead to find an additive value function compatible with the new preorder,
- searching over the relaxed domain $F \leq F^* + \eta$ could lead to an additive value function giving a preorder on $A^R$ sufficiently close to the reference preorder (in the sense of Kendall’s $\tau$).

5. On the UTA\textsuperscript{GMS} method

The preference information provided by the DM is similar to that of UTA, but the output is quite different (see Greco et al., in press).

In UTA\textsuperscript{GMS}, the preference information has the form of a partial preorder in a set of reference actions $A^R \subseteq A$ (i.e., a set of pairwise comparisons of reference actions). As a result, one obtains two rankings on set $A$, such that for any pair of actions $x, y \in A$,

(i) in the necessary ranking (partial preorder): $x$ is ranked at least as good as $y$ if and only if, $U(x) \geq U(y)$ for all the value functions compatible with the preference information provided by the DM;
(ii) in the possible ranking (strongly complete and negatively transitive relation): $x$ is ranked at least as good as $y$ if and only if, $U(x) \geq U(y)$ for at least one value function compatible with the preference information.
It should be noticed that a similar concept of necessary weak preference relation was already considered in Siskos (1982), where a so-called deterministic outranking relation \( S \) was defined as follows: \( a S b \) if and only if for every value function \( U \) from a finite set \( \mathcal{U} \), \( U(a) \geq U(b) \). Relation \( S \) corresponds to necessary weak preference relation \( \succeq^N \) with the following important difference:

- In (Siskos, 1982), the set \( \mathcal{U} \), called “utility system”, is composed of a finite number \( (2m) \) of value functions (“characteristic” of the polyhedron defined in the post-optimal analysis) which give a “satisfactory degree of consistency” between the weak order induced by the value functions and the weak order given by the DM.
- In UTA\textsuperscript{GMS} and in GRIP, the set \( \mathcal{U} \) is composed of all compatible value functions whose number is, in general, infinite and which are neither explicitly known nor calculated.

### 5.1. Main features

The main features of UTA\textsuperscript{GMS} are the following:

- It is using a general additive value function to represent preferences: a set of feasible value functions is identified and any additive function belonging to that set is called a compatible value function.
- The preference information can be given as a partial preorder on the set of reference actions.
- Two preference relations – necessary and possible – are considered to take into account certain or conceivable preferences, respectively.
- It can represent incomparability between actions: the necessary preference is not complete, in general.
- It provides robust conclusions: the necessary and possible preference relations are based on all compatible value functions, rather than on only one or few among the many possible functions, as it is usual in MCDA.
- It permits to detect inconsistent preference information: once the ordinal regression fails to find any compatible value function, the inconsistent pairwise comparisons can be detected to remove this impossibility.
- It can be used in an interactive procedure: the DM can modify the preference information verifying its impact on the preference relations in the set of considered actions.

### 5.2. The ordinal regression via linear programming

The preference information is given by the DM in the form of a partial preorder \( \succeq \) on the set of reference actions \( A^R \subseteq A \). A value function is called compatible if it is able to restore partial preorder \( \succeq \) on \( A^R \). Each compatible value function induces, moreover, a complete preorder on the whole set \( A \).

In particular, for any two actions \( x, y \in A \), a compatible value function orders \( x \) and \( y \) in one of the following ways: \( x \succeq y \), \( y \succeq x \), or \( x \sim y \). With respect to \( x, y \in A \), it is thus reasonable to ask the following two questions:

- Are \( x \) and \( y \) ordered in the same way by all compatible value functions?
- Is there at least one compatible value function ordering \( x \) at least as good as \( y \) (or \( y \) at least as good as \( x \))?

Having answers to these questions for all pairs of actions \( (x, y) \in A \times A \), one gets a necessary weak preference relation \( \succeq^N \) in \( A \), whose semantics is \( U(x) \geq U(y) \) for all compatible value functions, and a possible weak preference relation \( \succeq^p \) in \( A \), whose semantics is \( U(x) \geq U(y) \) for at least one compatible value function.

Let us remark that preference relations \( \succeq^N \) and \( \succeq^p \) are meaningful only if there exists at least one compatible value function. Observe also that in this case, for any \( a, b \in A^R \),

\[
a \succeq b \Rightarrow a \succeq^N b
\]

and

\[
a \succeq b \Rightarrow \text{not } (b \succeq^p a).
\]

In fact, if \( a \succeq b \), then for any compatible value function, \( U(a) \geq U(b) \) and, therefore, \( a \succeq^N b \). Moreover, if \( a \succeq b \), then for any compatible value function, \( U(a) > U(b) \) and, consequently, there is no compatible value function such that \( U(b) \geq U(a) \), which means not \( (b \succeq^p a) \).

Formally, a general additive compatible value function is an additive value function \( U(x) = \sum_{i=1}^n u_i(x_i) \) satisfying the following set of constraints:
\[ U(a) > U(b) \iff a \succ b \}\quad \forall a, b \in A^p \\
U(a) = U(b) \iff a \sim b \quad \{E^q\}, \]

\[ u_i(g_i(a_{i(1)})) - u_i(g_i(a_{i(j-1)})) \geq 0, \quad i = 1, \ldots, n, \quad j = 2, \ldots, m \\
u_i(g_i(a_{i(1)})) \geq 0, u_i(g_i(a_{i(m)})) \leq u_i(\beta_i), \quad i = 1, \ldots, n \\
u_i(x) = 0, \quad i = 1, \ldots, n \\
\sum_{i=1}^n u_i(\beta_i) = 1 \]

where \( \tau_i \) is the permutation on the set of indices of actions from \( A^p \) that reorders them according to the increasing evaluation on criterion \( g_i \), i.e.

\[ g_i(a_{i(1)}) \leq g_i(a_{i(2)}) \leq \cdots \leq g_i(a_{i(m-1)}) \leq g_i(a_{i(m)}). \]

Remark that, due to this formulation of the ordinal regression problem, no linear interpolation is required to express the marginal value of any reference action. Thus, one cannot expect that increasing the number of characteristic points will bring some “new” compatible additive value functions. In consequence, UTA\(^G\)MS considers all compatible additive value functions, while the classical UTA ordinal regression (4) deals with a subset of the whole set of compatible additive value functions, more precisely, the piecewise linear additive value functions relative to the considered characteristic points.

5.3. Computation of the relations \( \succeq^N \) and \( \succeq^p \)

In order to compute binary relations \( \succeq^p \) and \( \succeq^N \), UTA\(^G\)MS proceeds as follows. For all actions \( x, y \in A \), let \( \pi \) be a permutation of the indices of actions from set \( A^p \cup \{x, y\} \) that reorders them according to increasing evaluation on criterion \( g_i \), i.e.

\[ g_i(a_{\pi(1)}) \leq g_i(a_{\pi(2)}) \leq \cdots \leq g_i(a_{\pi(n-1)}) \leq g_i(a_{\pi(n)}), \]

where

- if \( A^p \cap \{x, y\} = \emptyset \), then \( \omega = m + 2 \),
- if \( A^p \cap \{x, y\} = \{x\} \) or \( A^p \cap \{x, y\} = \{y\} \), then \( \omega = m + 1 \),
- if \( A^p \cap \{x, y\} = \{x, y\} \), then \( \omega = m \).

Then, we can fix the characteristic points of \( u_i(g_i) \), \( i = 1, \ldots, n \), in

\[ g_i^0 = a_i, \quad g_i^j = g_i(a_{\pi(j)}) \quad \text{for } j = 1, \ldots, \omega, \quad g_i^{\omega+1} = \beta_i, \]

Let us consider the following set \( E(x, y) \) of ordinal regression constraints, with \( u_i(g_i^j) \), \( i = 1, \ldots, n \), \( j = 1, \ldots, \omega + 1 \), as variables:

\[ U(a) \geq U(b) + \varepsilon \iff a \succ b \}
\[ U(a) = U(b) \iff a \sim b \quad \forall a, b \in A^p \\
u_i(g_i^j) - u_i(g_i^{j-1}) \geq 0, \quad i = 1, \ldots, n, \quad j = 1, \ldots, \omega + 1 \\
u_i(g_i^n) = 0, \quad i = 1, \ldots, n \\
\sum_{i=1}^n u_i(g_i^{\omega+1}) = 1 \quad \{E(x, y)\}, \]

where \( \varepsilon \) is an arbitrarily small positive value, as in (4).

The above set of constraints depends on the pair of actions \( x, y \in A \) because their evaluations \( g_i(x) \) and \( g_i(y) \) give coordinates for two of (\( \omega + 1 \)) characteristic points of marginal value function \( u_i(x) \), for each \( i = 1, \ldots, n \). Note that for all \( x, y \in A \), \( E(x, y) = E(y, x) \).

Let us suppose that the polyhedron defined by the set of constraints \( E(x, y) \) is not empty. In this case we have that:

\[ x \succeq^N y \iff d(x, y) \geq 0, \]

where

\[ d(x, y) = \min \{ U(x) - U(y) \} \]

s.t. set \( E(x, y) \) of constraints
and

\[ x \succeq^{p} y \iff D(x, y) \geq 0, \]

where

\[
D(x, y) = \max \{ U(x) - U(y) \} \quad \text{s.t.} \quad \text{set } E(x, y) \text{ of constraints.}
\]

(7)

6. GRIP methodology

In this section we present a comprehensive description of the proposed GRIP methodology, including its main features, the preference information provided by the DM, the ordinal regression constraints and the fundamental properties of binary relations, the formulation of the linear programming problem, and a theoretical comparison with AHP and MACBETH methodologies.

6.1. Main features of GRIP

GRIP generalizes both UTA and UTA\textsuperscript{GMS} methods by adopting all features of UTA\textsuperscript{GMS} and taking into account additional preference information in the form of comparisons of intensities of preference between some pairs of reference actions. For actions \( x, y, w, z \in A \), these comparisons are expressed in two possible ways (not exclusive):

1. Comprehensively, on all criteria, like “\( x \) is preferred to \( y \) at least as much as \( w \) is preferred to \( z \)”.
2. Partially, on a particular criterion, like “\( x \) is preferred to \( y \) at least as much as \( w \) is preferred to \( z \), on criterion \( g_i \in F \)”.

6.2. The preference information provided by the decision maker

DM is expected to provide the following preference information:

- A partial preorder \( \succeq \) on \( A^k \), whose meaning is: for \( x, y \in A^k \)

\[ x \succeq y \iff \text{“} x \text{\ is at least as good as } y \text{\”}. \]

Moreover, \( \succ \) (preference) is the asymmetric part of \( \succeq \) and \( \sim \) (indifference) is the symmetric part given by \( \succeq \cap \succeq^{-1} \) (\( \succeq^{-1} \) is the inverse of \( \succeq \), i.e. for all \( x, y \in A^k \), \( x \succeq^{-1} y \iff y \succeq x \)).

- A partial preorder \( \succeq^* \) on \( A^k \times A^k \), whose meaning is: for \( x, y, w, z \in A^k \),

\[ (x, y) \succeq^* (w, z) \iff \text{“} x \text{\ is preferred to } y \text{\ at least as much as } w \text{\ is preferred to } z \text{\”}. \]

Also in this case, \( \succ^* \) is the asymmetric part of \( \succeq^* \) and \( \sim^* \) is the symmetric part given by \( \succeq^* \cap \succeq^*-1 \) (\( \succeq^*-1 \) is the inverse of \( \succeq^* \), i.e. for all \( x, y, w, z \in A^k \), \( (x, y) \succeq^*-1 (w, z) \iff (w, z) \succeq^* (x, y) \)).

- A partial preorder \( \succeq_i \) on \( A^k \times A^k \), whose meaning is: for \( x, y, w, z \in A^k \), \( (x, y) \succeq_i (w, z) \iff \text{“} x \text{\ is preferred to } y \text{\ at least as much as } w \text{\ is preferred to } z \text{\”} \) on criterion \( g_i \), \( i \in I \).

The intensities of preferences can also be expressed in terms of pre-defined degrees of intensity of preference corresponding to some semantic categories, e.g., the preference of \( x \) over \( y \) is “moderate”, the preference of \( w \) over \( z \) is “very strong”, etc. This way of handling the intensities of preferences is typical for MACBETH-like procedures (Bana e Costa and Vansnick, 1994; Bana e Costa et al., 2005).

In the following, we also consider the weak preference relation \( \preceq_i \) being a complete preorder whose meaning is: for all \( x, y \in A \),

\[ x \preceq_i y \iff \text{“} x \text{\ is at least as good as } y \text{\”} \] on criterion \( g_i \), \( i \in I \).

Weak preference relations \( \preceq_i, i \in I \), are not provided by the DM, but they are obtained directly from the evaluations of actions \( x \) and \( y \) on criteria \( g_i \), i.e., \( x \preceq_i y \iff g_i(x) \geq g_i(y), i \in I \).

6.3. Ordinal regression constraints and properties of binary relations

In this subsection, we present a set of the ordinal regression constraints that interpret the preference information in terms of conditions on the compatible value functions. After that, we give fundamental properties of six binary relations resulting from the consideration of the set of compatible value functions. These relations are core relations of the method proposed in this paper.
6.3.1. Ordinal regression constraints

The value function $U : A \rightarrow [0, 1]$ should satisfy the following constraints corresponding to DM’s preference information:

(a) $U(w) > U(z)$ if $w \succ z$,
(b) $U(w) = U(z)$ if $w \sim z$,
(c) $U(w) - U(z) > U(r) - U(s)$ if $(w, z) \succ^1 (r, s)$,
(d) $U(w) - U(z) = U(r) - U(s)$ if $(w, z) \sim (r, s)$,
(e) $u_i(w) \geq u_i(z)$ if $w \succeq_i z$, $i \in I$,
(f) $u_i(w) - u_i(z) > u_i(r) - u_i(s)$ if $(w, z) \succ^1_i (r, s)$, $i \in I$,
(g) $u_i(w) - u_i(z) = u_i(r) - u_i(s)$ if $(w, z) \equiv^1_i (r, s)$, $i \in I$.

Let us remark that within UTA-like methods, constraint (a) is written as $U(w) \geq U(z) + \varepsilon$, where $\varepsilon > 0$ is a threshold exogenously introduced. Analogously, constraints (c) and (f) should be written as,

$$U(w) - U(z) \geq U(r) - U(s) + \varepsilon$$

and

$$u_i(w) - u_i(z) \geq u_i(r) - u_i(s) + \varepsilon.$$

However, we would like to avoid the use of any exogenous parameter and, therefore, instead of setting an arbitrary value of $\varepsilon$, we consider it as an auxiliary variable, and we test the feasibility of constraints (a), (c), and (f) (see Section 6.4). In this way, we take into account all possible value functions, even those for which threshold $\varepsilon$ is very small. This way is also safer from the viewpoint of “objectivity” of the whole methodology. In fact, the value of $\varepsilon$ is not meaningful by itself and it is useful only because it permits to discriminate preference from indifference.

Moreover, the following normalization constraints should also be taken into account:

(h) $u_i(x^*_i) = 0$, where $x^*_i$ is such that $x^*_i = \min \{g_i(x) : x \in A\}$,
(i) $\sum_{i \in I} u_i(x^*_i) = 1$, where $y^*_i$ is such that $y^*_i = \max \{g_i(x) : x \in A\}$.

If the constraints from (a) to (i) are satisfied, then the partial preorders $\succeq$ and $\succ^1$ on $A^8$ can be extended on $A$ in two different ways as follows:

(1) Through the choice of one value function $U$ considered the “best” among all the compatible value functions $U$ satisfying constraints from (a) to (i) and setting, for all $x, y, w, z \in A$,

- $(x, y) \succeq U(x) \geq U(y)$, and
- $(x, y) \succ^1 (w, z)$ if $U(x) - U(y) \geq U(w) - U(z)$

(for a survey of ways of choosing the “best” value function consistent with preference information, see (Jacquet-Lagréa and Siskos, 1982; Siskos et al., 2005)).

(2) Through the identification of two weak preference relations $\succeq_N$ and $\succeq_P$ and two binary relations comparing intensity of preference $\succeq^N$ and $\succeq^P$ (Greco et al., 2003, in press), called necessary ($N$) and possible ($P$), respectively:

(a) For each, $x, y \in A$, $x \succeq_N y$ (“$x$ is necessarily at least as good as $y$”) means that $U(x) \geq U(y)$ for all compatible value functions $U$.

The following holds,

$x \succeq_N y \iff \inf \{U(x) - U(y)\} \geq 0$, 

where the infimum is calculated on the set of value functions satisfying constraints from (a) to (i).

(b) For each, $x, y \in A$, $x \succeq_P y$ (“$x$ is possibly at least as good as $y$”) means that $U(x) \geq U(y)$ for at least one compatible value function $U$.

The following holds,

$x \succeq_P y \iff \inf \{U(y) - U(x)\} \leq 0$, 

where the infimum is calculated on the set of value functions satisfying constraints from (a) to (i).

Remark: Observe that $\inf \{U(y) - U(x)\} \leq 0$ means that it is false that for all $U$, $U(y) - U(x) > 0$, i.e. there exists at least one $U$ for which $U(y) - U(x) \leq 0$ or, equivalently, $U(x) - U(y) \geq 0$. Observe that $\sup \{U(x) - U(y)\} \geq 0$ is not equivalent to $\inf \{U(y) - U(x)\} \leq 0$. In fact, we can have $\sup \{U(x) - U(y)\} \geq 0$ also in case where for all $U$, $U(x) - U(y) < 0$. For example, if the value of $U(x) - U(y) \in [a, 0]$, where $a$ is any negative number, we have that $U(x) - U(y) < 0$ always and, nevertheless, $\sup \{U(x) - U(y)\} \geq 0$. 

6.3.2. Fundamental properties of necessary and possible binary relations

The following theorem presents some basic properties concerning binary relations \( \preceq_N, \preceq_P, \preceq^N, \) and \( \preceq^P \).

**Theorem 6.1.** If constraints (a)–(i) are satisfied, then the following properties hold:

1. for all \( x, y \in A \), \( x \preceq^N y \) if and only if \( y \preceq^P x \) (Greco et al., in press);
2. for all \( x, y \in A \), \( x \preceq_N y \) if and only if \( y \preceq^P x \) (Greco et al., in press);
3. \( \preceq^N \) is a partial preorder (i.e., the relation is transitive and reflexive) and \( \preceq^P \) is strongly complete (i.e., for each \( x, y \in A \) at least one of the following two relations is true, \( x \preceq^N y \) or \( y \preceq^N x \)) and negatively transitive (i.e., for each \( x, y, z \in A \), if not \( x \preceq_N y \) and not \( y \preceq_N z \), then not \( x \preceq_N z \) (Greco et al., in press);
4. for all \( x, y \in A \), \( x \preceq_N y \) or \( y \preceq_P x \) (Greco et al., 2003, in press);
5. for all \( x, y, z \in A \), \( x \preceq_N y \) or \( y \preceq_N z \) (Greco et al., 2003, in press);
6. for all \( x, y, z \in A \), \( x \preceq_P y \) or \( y \preceq_P z \) (Greco et al., 2003, in press);
7. for all \( x, y, w, z \in A \), \( x \preceq^N y \) or \( y \preceq^N z \) (Greco et al., 2003, in press);
8. for all \( x, y, w, z \in A \), \( x \preceq_N y \) or \( y \preceq_P z \) (Greco et al., 2003, in press);
9. \( \preceq^N \) is a partial preorder and \( \preceq^P \) is strongly complete and negatively transitive;
10. for all \( x, y \in A \), \( x \preceq_N y \) if and only if \( y \preceq_N x \) (Greco et al., 2003, in press);
11. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) and \( y \preceq_N z \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press);
12. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) and \( y \preceq_N z \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press);
13. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press);
14. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press);
15. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press);
16. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press);
17. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press);
18. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press);
19. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press);
20. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press);
21. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press);
22. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press);
23. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press);
24. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press);
25. for all \( x, y, z, w, r, s \in A \), \( x \preceq_N y \) if and only if \( x \preceq_N z \) (Greco et al., 2003, in press).
for all \(x, x', y \in A\), \((x', y) \succsim_N (x, y) \iff x' \succsim_N x; \)

(27) for all \(x, y, y' \in A\), \((x, y') \succsim_N (x, y) \iff y' \succsim_N y; \)

(28) for all \(x, y, y' \in A\), \((x, y') \succsim_N (x, y) \iff y' \succsim_N y; \)

(29) \(\succsim_N\) is a preorder and \(\succsim_p\) is strongly complete and negatively transitive, for all \(i \in I; \)

(30) for all \(x, y, w, z \in A\), \((x, y) \succsim_N (w, z) \text{ or } (w, z) \succsim_N (x, y), \text{ for all } i \in I; \)

(31) for all \(x, y, z \in A\), \((x, y) \succsim_N (z, x) \text{ and } (z, x) \succsim_N (y, z) \text{ for all } i \in I; \)

(32) for all \(x, y, w, z \in A\), \((x, y) \succsim_N (w, z) \text{ and } (w, z) \succsim_N (x, y), \text{ for all } i \in I; \)

(33) for all \(x, y, w, z \in A\), \((x, y) \succsim_N (w, z) \Rightarrow (x, y) \succsim_N (w, z), \text{ for all } i \in I; \)

(34) for all \(x, y, w, z \in A\), \((x, y) \succsim_N (w, z) \Rightarrow (x, y) \succsim_N (w, z), \text{ for all } i \in I; \)

(35) for all \(x, y, w, z \in A\), \((x, y) \succsim_N (w, z) \Rightarrow (x, y) \succsim_N (w, z), \text{ for all } i \in I; \)

(36) for all \(x, y, w, z \in A\), \((x, y) \succsim_N (w, z) \Rightarrow (x, y) \succsim_N (w, z), \text{ for all } i \in I; \)

(37) for all \(x, y, w, z \in A\), \((x, y) \succsim_N (w, z) \Rightarrow (x, y) \succsim_N (w, z), \text{ for all } i \in I; \)

(38) for all \(x, y, w, z \in A\), \((x, y) \succsim_N (w, z) \Rightarrow (x, y) \succsim_N (w, z), \text{ for all } i \in I; \)

(39) for all \(x, y, w, z \in A\), \((x, y) \succsim_N (w, z) \Rightarrow (x, y) \succsim_N (w, z), \text{ for all } i \in I; \)

(40) for all \(x, y, w, z \in A\), \((x, y) \succsim_N (w, z) \Rightarrow (x, y) \succsim_N (w, z), \text{ for all } i \in I; \)

(41) for all \(x, y, x', y' \in A\), \((x', y') \succsim_N (x, y) \Rightarrow (x', y') \succsim_N (x, y), \text{ for all } i \in I; \)

(42) for all \(x, y, x', y' \in A\), \((x', y') \succsim_N (x, y) \Rightarrow (x', y') \succsim_N (x, y), \text{ for all } i \in I; \)

(43) for all \(x, y, x', y' \in A\), \((x', y') \succsim_N (x, y) \Rightarrow (x', y') \succsim_N (x, y), \text{ for all } i \in I; \)

(44) for all \(x, y, x', y' \in A\), \((x', y') \succsim_N (x, y) \Rightarrow (x', y') \succsim_N (x, y), \text{ for all } i \in I; \)

The proof of this theorem is provided in the Appendix. Now, let us explain the contents of the theorem with respect to their relevance from the point of view of MCDA:

1. it corresponds to the general idea that if a result is necessary, then it must also be possible: within MCDA this implies that if a weak preference is necessary, then it is possible too (Greco et al., 2003, in press);
2. since we are considering a set of value functions compatible with the preferences expressed by the DM, then if the weak preference of \(x\) over \(y\) is specified by the DM, then each compatible value function must represent the weak preference of \(x\) over \(y\), which is thus necessary (and, for (1), possible too) (Greco et al., 2003, in press);
3. it expresses the preference structure of \(\succsim_N\) and \(\succsim_p\), that is \(\succsim_N\) is a preorder (i.e. the relation is transitive and reflexive), and \(\succsim_p\) is strongly complete and negatively transitive (Greco et al., 2003, in press);
4. it expresses a completeness condition for \(\succsim_N\) and \(\succsim_p\), that is for all \(x, y \in A, x \succsim_N y \text{ or } y \succsim_N x\) (Greco et al., 2003, in press);
5. it expresses a transitivity condition for \(\succsim_N\) and \(\succsim_p\), that is for all \(x, y, z \in A, x \succsim_N y \text{ and } y \succsim_N z \Rightarrow x \succsim_N z\); 
6. it expresses a transitivity condition for \(\succsim_N\) and \(\succsim_p\), that is for all \(x, y, z \in A, x \succsim_N y \text{ and } y \succsim_N z \Rightarrow x \succsim_N z\); 
7. analogously from (1), it expresses the general idea that if a result is necessary, then it must also be possible; differently from (1), it predicated this principle for \(\succsim_N\) and \(\succsim_p\), that is, if it is necessary that the intensity of preference of \(x\) over \(y\) is not smaller than the intensity of preference of \(w\) over \(z\), then it is possible too;
8. analogously to (2), since we are considering a set of value functions compatible with the preferences expressed by the DM, then if the fact that the intensity of preference of \(x\) over \(y\) is not smaller than the intensity of preference of \(w\) over \(z\) has been expressed by the DM, then each compatible value function must represent this fact, which is thus necessary (and, for (7), possible too);
9. analogously to (3), it expresses the preference structure of \(\succsim_N\) and \(\succsim_p\), that is \(\succsim_N\) is a partial preorder and \(\succsim_p\) is strongly complete and negatively transitive;
10. analogously to (4), it expresses a completeness condition; differently from (4), it expresses this condition for \(\succsim_N\) and \(\succsim_p\), that is for all \(x, y, w, z \in A, (x, y) \succsim_N (w, z) \text{ or } (w, z) \succsim_N (x, y); \)
11. analogously to (5), it expresses a transitivity condition; differently from (5) it expresses this condition for \(\succsim_N\) and \(\succsim_p\), that is for all \(x, y, w, z, r, s \in A, (x, y) \succsim_N (w, z) \text{ and } (w, z) \succsim_N (r, s) \Rightarrow (x, y) \succsim_N (r, s); \)
12. analogously to (6), it expresses a transitivity condition; differently from (6) it expresses this condition for \(\succsim_N\) and \(\succsim_p\), that is for all \(x, y, w, z, r, s \in A, (x, y) \succsim_N (w, z) \text{ and } (w, z) \succsim_N (r, s) \Rightarrow (x, y) \succsim_N (r, s); \)
13. it states a property relating \(\succsim_N\) and \(\succsim_p\); if the intensity of preference of \(x\) over \(y\) is necessarily not smaller than the intensity of preference of \(w\) over \(z\) (i.e. \((x, y) \succsim_N (w, z)\)) and we replace \(x\) by \(x'\), with \(x'\) necessarily at least as good as \(x\) (i.e. \(x' \succsim_N x\)), then the intensity of preference of \(x'\) over \(y\) is necessarily not smaller than the intensity of preference of \(w\) over \(z\) (i.e. \((x', y) \succsim_N (w, z)\)); 
14. it states a property relating \(\succsim_N\) and \(\succsim_p\); if the intensity of preference of \(x\) over \(y\) is possibly not smaller than the intensity of preference of \(w\) over \(z\) (i.e. \((x, y) \succsim_p (w, z)\)) and we replace \(x\) by \(x'\), with \(x'\) necessarily at least as good as \(x\) (i.e. \(x' \succsim_N x\)), then the intensity of preference of \(x'\) over \(y\) is possibly not smaller than the intensity of preference of \(w\) over \(z\) (i.e. \((x', y) \succsim_p (w, z)\));
(15) it states a property relating $\succeq^p$, $\succeq^w$ and $\succeq^e$; if the intensity of preference of $x$ over $y$ is necessarily not smaller than the intensity of preference of $w$ over $z$ (i.e. $(x,y) \succeq^w (w,z)$) and we replace $x$ by $x'$, with $x'$ possibly at least as good as $x$ (i.e. $x' \succeq^p x$), then the intensity of preference of $x'$ over $y$ is possibly not smaller than the intensity of preference of $w$ over $z$ (i.e. $(x',y) \succeq^e (w,z)$);

(16) analogously to (13), it states a property relating $\succeq^N$ and $\preceq^w$; if the intensity of preference of $x$ over $y$ is necessarily not smaller than the intensity of preference of $w$ over $z$ (i.e. $(x,y) \succeq^w (w,z)$) and, differently from (13), we replace $y$ by $y'$, with $y$ necessarily at least as good as $y'$ (i.e. $y \succeq^N y'$), then the intensity of preference of $x$ over $y'$ is necessarily not smaller than the intensity of preference of $w$ over $z$ (i.e. $(x,y') \succeq^w (w,z)$);

(17) analogously to (14), it states a property relating $\succeq^N$ and $\succeq^e$; if the intensity of preference of $x$ over $y$ is possibly not smaller than the intensity of preference of $w$ over $z$ (i.e. $(x,y) \succeq^w (w,z)$) and, differently from (14), we replace $y$ by $y'$, with $y$ necessarily at least as good as $y'$ (i.e. $y \succeq^N y'$), then the intensity of preference of $x$ over $y'$ is possibly not smaller than the intensity of preference of $w$ over $z$ (i.e. $(x,y') \succeq^w (w,z)$);

(18) analogously to (15), it states a property relating $\succeq^e$, $\preceq^N$ and $\succeq^e$; if the intensity of preference of $x$ over $y$ is necessarily not smaller than the intensity of preference of $w$ over $z$ (i.e. $(x,y) \succeq^e (w,z)$) and, differently from (15), we replace $y$ by $y'$, with $y$ possibly at least as good as $y'$ (i.e. $y \succeq^e y'$), then the intensity of preference of $x$ over $y'$ is possibly not smaller than the intensity of preference of $w$ over $z$ (i.e. $(x,y') \succeq^e (w,z)$);

(19) analogously to (13) and (16), it states a property relating $\succeq^N$ and $\preceq^e$; if the intensity of preference of $x$ over $y$ is necessarily not smaller than the intensity of preference of $w$ over $z$ (i.e. $(x,y) \succeq^w (w,z)$) and, differently from (13) and (16), we replace $w$ by $w'$, with $w$ necessarily at least as good as $w'$ (i.e. $w \succeq^N w'$), then the intensity of preference of $x$ over $y$ is necessarily not smaller than the intensity of preference of $w'$ over $z$ (i.e. $(x,y) \succeq^w (w',z)$);

(20) analogously to (14) and (17), it states a property relating $\preceq^e$ and $\preceq^w$; if the intensity of preference of $x$ over $y$ is possibly not smaller than the intensity of preference of $w$ over $z$ (i.e. $(x,y) \succeq^w (w,z)$) and, differently from (14) and (17), we replace $w$ by $w'$, with $w$ necessarily at least as good as $w'$ (i.e. $w \succeq^N w'$), then the intensity of preference of $x$ over $y$ is possibly not smaller than the intensity of preference of $w'$ over $z$ (i.e. $(x,y) \succeq^w (w',z)$);

(21) analogously to (15) and (18), it states a property relating $\succeq^N$ and $\succeq^e$; if the intensity of preference of $x$ over $y$ is necessarily not smaller than the intensity of preference of $w$ over $z$ (i.e. $(x,y) \succeq^w (w,z)$) and, differently from (15) and (18), we replace $w$ by $w'$, with $w$ possibly at least as good as $w'$ (i.e. $w \succeq^N w'$), then the intensity of preference of $x$ over $y$ is possibly not smaller than the intensity of preference of $w'$ over $z$ (i.e. $(x,y) \succeq^w (w',z)$);

(22) analogously to (13), (16) and (19), it states a property relating $\succeq^N$ and $\preceq^e$; if the intensity of preference of $x$ over $y$ is necessarily not smaller than the intensity of preference of $w$ over $z$ (i.e. $(x,y) \succeq^w (w,z)$) and, differently from (13), (16) and (19), we replace $z$ by $z'$, with $z'$ necessarily at least as good as $z$ (i.e. $z \succeq^N z'$), then the intensity of preference of $x$ over $y$ is necessarily not smaller than the intensity of preference of $w$ over $z'$ (i.e. $(x,y) \succeq^w (w,z')$);

(23) analogously to (14), (17) and (20), it states a property relating $\preceq^e$ and $\preceq^w$; if the intensity of preference of $x$ over $y$ is possibly not smaller than the intensity of preference of $w$ over $z$ (i.e. $(x,y) \succeq^w (w,z)$) and, differently from (14), (17) and (20), we replace $z$ by $z'$, with $z'$ necessarily at least as good as $z$ (i.e. $z \succeq^N z'$), then the intensity of preference of $x$ over $y$ is necessarily not smaller than the intensity of preference of $w$ over $z'$ (i.e. $(x,y) \succeq^w (w,z')$);

(24) analogously to (15), (18) and (21), it states a property relating $\succeq^p$, $\succeq^N$ and $\succeq^e$; if the intensity of preference of $x$ over $y$ is necessarily not smaller than the intensity of preference of $w$ over $z$ (i.e. $(x,y) \succeq^w (w,z)$) and, differently from (15), (18) and (21), we replace $z$ by $z'$, with $z'$ possibly at least as good as $z$ (i.e. $z \succeq^N z'$), then the intensity of preference of $x$ over $y$ is possibly not smaller than the intensity of preference of $w$ over $z'$ (i.e. $(x,y) \succeq^w (w,z')$);

(25) it relates $\preceq^e$ and $\preceq^N$; if we replace $x$ by $x'$, then the intensity of preference of $x$ over $y$ is necessarily not smaller than the intensity of preference of $x$ over $y$ (i.e. $(x',y) \succeq^N (x,y)$), if and only if $x'$ is necessarily at least as good as $x$ (i.e. $x' \succeq^N x$);

(26) it relates $\preceq^e$ and $\preceq^p$ in a way analogous to the one in which (25) relates $\preceq^e$ and $\preceq^N$; if we replace $x$ by $x'$, then the intensity of preference of $x$ over $y$ is possibly not smaller than the intensity of preference of $x$ over $y$ (i.e. $(x',y) \succeq^p (x,y)$), if and only if $x'$ is possibly at least as good as $x$ (i.e. $x' \succeq^p x$);

(27) similarly to (25), it relates $\preceq^N$ and $\preceq^N$; if, differently form (25), we replace $y$ by $y'$, then the intensity of preference of $x$ over $y$ is necessarily not smaller than the intensity of preference of $x$ over $y'$ (i.e. $(x,y) \succeq^N (x,y')$), if and only if $y'$ is necessarily at least as good as $y$ (i.e. $y' \succeq^N y$);

(28) it relates $\preceq^e$ and $\preceq^e$ in a way analogous to the one in which (27) relates $\preceq^N$ and $\preceq^N$; if we replace $y$ by $y'$, then the intensity of preference of $x$ over $y$ is possibly not smaller than the intensity of preference of $x$ over $y'$ (i.e. $(x,y) \succeq^e (x,y')$), if and only if $y'$ is possibly at least as good as $y$ (i.e. $y' \succeq^e y$);

(29) it expresses the preference structures of $\preceq^i$, $\preceq^i$, $i \in I$; $\preceq^i$ is a partial preorder and $\preceq^i$ is strongly complete and negatively transitive;

(30) it expresses a completeness condition for $\preceq^i$ and $\preceq^i$, that is for all $x,y,w,z \in A$, $(x,y) \preceq^i (w,z)$ or $(w,z) \preceq^i (x,y)$, for all $i \in I$;
(31) it expresses a transitivity condition for \( \succsim_i^x \) and \( \succsim_i^y \), for all \( i \in I \), that is, for all \( x,y,w,z,r,s \in A \),
\[
[x,y] \succsim_i^x(w,z) \land (w,z) \succsim_i^y(r,s) \implies (x,y) \succsim_i^y(r,s);
\]
(32) it expresses a transitivity condition for \( \succsim_i^x \) and \( \succsim_i^y \), for all \( i \in I \), that is, for all \( x,y,w,z,r,s \in A \),
\[
[x,y] \succsim_i^x(w,z) \land (w,z) \succsim_i^y(r,s) \implies (x,y) \succsim_i^y(r,s);
\]
(33) it relates the order between evaluations on criterion \( g_i \) and \( \succsim_i^y \), for all \( i \in I \); for all \( x,x',y,w,z \in A \), if the intensity of preference of \( x \) over \( y \) with respect to criterion \( g_i \) is necessarily not smaller than that of \( w \) over \( z \) (i.e. \( (x,y) \succsim_i^y(w,z) \)), and we replace \( x \) by \( x' \) with \( g_i(x') \geq g_i(x) \), then also the intensity of preference of \( x' \) over \( y \) with respect to criterion \( g_i \) is necessarily not smaller than that of \( w \) over \( z \) (i.e. \( (x',y) \succsim_i^y(w,z) \));
(34) it relates the order between evaluations on criterion \( g_i \) and \( \succsim_i^y \), for all \( i \in I \); for all \( x,x',y,w,z \in A \), if the intensity of preference of \( x \) over \( y \) with respect to criterion \( g_i \) is possibly not smaller than that of \( w \) over \( z \) (i.e. \( (x,y) \succsim_i^y(w,z) \)), and we replace \( x \) by \( x' \) with \( g_i(x') \geq g_i(x) \), then also the intensity of preference of \( x' \) over \( y \) with respect to criterion \( g_i \) is possibly not smaller than that of \( w \) over \( z \) (i.e. \( (x',y) \succsim_i^y(w,z) \));
(35) it relates the order between evaluations on criterion \( g_i \) and \( \succsim_i^y \), for all \( i \in I \); for all \( x,y,x',w,z \in A \), if the intensity of preference of \( x \) over \( y \) with respect to criterion \( g_i \) is necessarily not smaller than that of \( w \) over \( z \) (i.e. \( (x,y) \succsim_i^y(w,z) \)), and we replace \( y \) by \( y' \) with \( g_i(y) \geq g_i(y') \), then also the intensity of preference of \( x \) over \( y' \) with respect to criterion \( g_i \) is necessarily not smaller than that of \( w \) over \( z \) (i.e. \( (x,y') \succsim_i^y(w,z) \));
(36) it relates the order between evaluations on criterion \( g_i \) and \( \succsim_i^y \), for all \( i \in I \); for all \( x,y,x',w,z \in A \), if the intensity of preference of \( x \) over \( y \) with respect to criterion \( g_i \) is possibly not smaller than that of \( w \) over \( z \) (i.e. \( (x,y) \succsim_i^y(w,z) \)), and we replace \( y \) by \( y' \) with \( g_i(y) \geq g_i(y') \), then also the intensity of preference of \( x \) over \( y' \) with respect to criterion \( g_i \) is possibly not smaller than that of \( w \) over \( z \) (i.e. \( (x,y') \succsim_i^y(w,z) \));
(37) it relates the order between evaluations on criterion \( g_i \) and \( \succsim_i^y \), for all \( i \in I \); for all \( x,y,y',w,z \in A \), if the intensity of preference of \( x \) over \( y \) with respect to criterion \( g_i \) is necessarily not smaller than that of \( w \) over \( z \) (i.e. \( (x,y) \succsim_i^y(w,z) \)), and we replace \( w \) by \( w' \) with \( g_i(w) \geq g_i(w') \), then also the intensity of preference of \( x \) over \( y' \) with respect to criterion \( g_i \) is necessarily not smaller than that of \( w' \) over \( z \) (i.e. \( (x,y') \succsim_i^y(w',z) \));
(38) it relates the order between evaluations on criterion \( g_i \) and \( \succsim_i^y \), for all \( i \in I \); for all \( x,y,y',w,z \in A \), if the intensity of preference of \( x \) over \( y \) with respect to criterion \( g_i \) is possibly not smaller than that of \( w \) over \( z \) (i.e. \( (x,y) \succsim_i^y(w,z) \)), and we replace \( w \) by \( w' \) with \( g_i(w) \geq g_i(w') \), then also the intensity of preference of \( x \) over \( y' \) with respect to criterion \( g_i \) is possibly not smaller than that of \( w' \) over \( z \) (i.e. \( (x,y') \succsim_i^y(w',z) \));
(39) it relates the order between evaluations on criterion \( g_i \) and \( \succsim_i^y \), for all \( i \in I \); for all \( x,y,y',z' \in A \), if the intensity of preference of \( x \) over \( y \) with respect to criterion \( g_i \) is necessarily not smaller than that of \( w \) over \( z \) (i.e. \( (x,y) \succsim_i^y(w,z) \)), and we replace \( z \) by \( z' \) with \( g_i(z') \geq g_i(z) \), then also the intensity of preference of \( x \) over \( y \) with respect to criterion \( g_i \) is necessarily not smaller than that of \( w \) over \( z \) (i.e. \( (x,y) \succsim_i^y(w,z) \));
(40) it relates the order between evaluations on criterion \( g_i \) and \( \succsim_i^y \), for all \( i \in I \); for all \( x,y,y',z' \in A \), if the intensity of preference of \( x \) over \( y \) with respect to criterion \( g_i \) is possibly not smaller than that of \( w \) over \( z \) (i.e. \( (x,y) \succsim_i^y(w,z) \)), and we replace \( z \) by \( z' \) with \( g_i(z') \geq g_i(z) \), then also the intensity of preference of \( x \) over \( y \) with respect to criterion \( g_i \) is possibly not smaller than that of \( w \) over \( z \) (i.e. \( (x,y) \succsim_i^y(w,z) \));
(41) it states a property relating the order between evaluations on criterion \( g_i \) and \( \succsim_i^y \); for all \( x,x',y \in A \), if \( g_i(x') \geq g_i(x) \), then, with respect to criterion \( g_i \), the intensity of preference of \( x' \) over \( y \) is necessarily not smaller than the intensity of preference of \( x \) over \( y \) (i.e. \( (x',y) \succsim_i^y(x,y) \));
(42) it states a property relating \( \succsim_i^y \) and the order between evaluations on criterion \( g_i \); for all \( x,x',y \in A \), if the intensity of preference of \( x' \) over \( y \) is possibly greater than the intensity of preference of \( x \) over \( y \) with respect to criterion \( g_i \), then \( g_i(x') \geq g_i(x) \);
(43) it states a property relating the order between evaluations on criterion \( g_i \) and \( \succsim_i^y \); for all \( x,x,y',z \in A \), if \( g_i(y') \geq g_i(y) \), then, with respect to criterion \( g_i \), the intensity of preference of \( x' \) over \( y \) is necessarily not smaller than the intensity of preference of \( x \) over \( y \) (i.e. \( (x,y') \succsim_i^y(x,y) \));
(44) it states a property relating \( \succsim_i^y \) and the order between evaluations on criterion \( g_i \); for all \( x,y,y' \in A \), if the intensity of preference of \( x \) over \( y \) is possibly greater than the intensity of preference of \( x \) over \( y' \) with respect to criterion \( g_i \), then \( g_i(y') \geq g_i(y) \).

6.4. Computational issues

In order to conclude the truth or falsity of binary relations \( \succsim^N,\succsim^P,\succsim^x,\succsim^y,\succsim^x,\succsim^y \) and \( \succsim^r \), we have to take into account that, for all \( x,y,w,z \in A \) and \( i \in I \):

1. \( x \succsim^N y \iff \inf \{ U(x) - U(y) \} \geq 0 \),
2. \( x \succsim^y y \iff \inf \{ U(y) - U(x) \} \leq 0 \),
(3) \((x, y) \succeq^p (w, z) \iff \inf \{ (U(x) - U(y)) - (U(w) - U(z)) \} \geq 0,\)
(4) \((x, y) \succeq^p (w, z) \iff \inf \{ (U(w) - U(z)) - (U(x) - U(y)) \} \leq 0,\)
(5) \((x, y) \succeq^p (w, z) \iff \inf \{ (u_i(x_i) - u_i(y_i)) - (u_i(w_i) - u_i(z_i)) \} \geq 0,\)
(6) \((x, y) \succeq^p (w, z) \iff \inf \{ (u_i(w_i) - u_i(z_i)) - (u_i(x_i) - u_i(y_i)) \} \leq 0.\)

with the infimum calculated on the set of value functions satisfying constraints from (a) to (i). Let us remark, however, that the linear programming is not able to handle strict inequalities such as the above (a), (c), and (f). Moreover, linear programming permits to calculate the minimum or the maximum of an objective function and not an infimum. Therefore, to use linear programming for testing the truth of binary relations \(\succeq^N, \succeq^p, \succeq^v, \succeq^i, \succeq^v\), we need to reformulate properly the above properties from (1) to (6). With this aim, the following result (see Marichal and Roubens, 2000) can be taken into account (the adopted notation is used only locally in this proposition and does not inherit the meaning from other parts of the paper).

**Proposition 6.1.** \(x \in \mathbb{R}\) is a solution of the linear system,

\[
\begin{aligned}
\sum_{j=1}^{n} a_{ij} x_j &\geq b_i, & i = 1, \ldots, p, \\
\sum_{j=1}^{n} c_{ij} x_j &\geq d_i, & i = 1, \ldots, q,
\end{aligned}
\]

iff there exists \(\varepsilon > 0\), such that

\[
\begin{aligned}
\sum_{j=1}^{n} a_{ij} x_j &\geq b_i, & i = 1, \ldots, p, \\
\sum_{j=1}^{n} c_{ij} x_j &\geq d_i + \varepsilon, & i = 1, \ldots, q.
\end{aligned}
\]

In particular, a solution exists iff the following linear programming:

\[
\text{max } \varepsilon
\]

subject to:

\[
\begin{aligned}
\sum_{j=1}^{n} a_{ij} x_j &\geq b_i, & i = 1, \ldots, p, \\
\sum_{j=1}^{n} c_{ij} x_j &\geq d_i + \varepsilon, & i = 1, \ldots, q
\end{aligned}
\]

has an optimal solution \((x^*, \varepsilon^*)\) with an optimal value \(\varepsilon^* > 0\). In this case, \(x^*\) is a solution of the first system.

In order to use Proposition 5.1, one must first reformulate the constraints (a), (c) and (f) as follows:

(a') \(U(x) \succeq U(y) + \varepsilon\) if \(x \succ y;\)
(c') \(U(x) - U(y) \succeq U(w) - U(z) + \varepsilon\) if \((x, y) \succ^+ (w, z);\)
(f') \(u_i(x) - u_i(y) \succeq u_i(w) - u_i(z) + \varepsilon\) if \((x, y) \succ^+ (w, z).\)

Notice that constraints (a), (c) and (f) are equivalent to (a'), (c'), and (f') whenever \(\varepsilon > 0.\)

Then, properties (1)–(6) have to be reformulated such that the search of the infimum is replaced by the calculation of the maximum value of \(\varepsilon\) on the set of value functions satisfying constraints from (a) to (i), with constraints (a), (c), and (f) transformed to (a'), (c'), and (f'), plus constraints specific for each property:

(1') \(x \succeq^p y \iff \varepsilon^* > 0, \text{ where } \varepsilon^* = \max \varepsilon, \text{ subject to the constraints (a'), (b), (c'), (d), (e), (f')}, \text{ plus the constraint } U(x) \succeq U(y);\)
(2') \(x \succeq^N y \iff \varepsilon^* < 0, \text{ where } \varepsilon^* = \max \varepsilon, \text{ subject to the constraints (a'), (b), (c'), (d), (e), (f')}, \text{ plus the constraint } U(y) \succeq U(x) + \varepsilon;\)
(3') \((x, y) \succeq^o (w, z) \iff \varepsilon^* > 0, \text{ where } \varepsilon^* = \max \varepsilon, \text{ subject to the constraints (a'), (b), (c'), (d), (e), (f')}, \text{ plus the constraint } (U(x) - U(y)) - (U(w) - U(z)) \geq 0;\)
(4') \((x, y) \succeq^v (w, z) \iff \varepsilon^* \leq 0, \text{ where } \varepsilon^* = \max \varepsilon, \text{ subject to the constraints (a'), (b), (c'), (d), (e), (f')}, \text{ plus the constraint } (U(w) - U(z)) - (U(x) - U(y)) \geq \varepsilon;\)
(5') \((x, y) \succeq^i (w, z) \iff \varepsilon^* > 0, \text{ where } \varepsilon^* = \max \varepsilon, \text{ subject to the constraints (a'), (b), (c'), (d), (e), (f')}, \text{ plus the constraint } (u_i(x_i) - u_i(y_i)) - (u_i(w_i) - u_i(z_i)) \geq 0;\)
(6') \((x, y) \succ^V (w, z) \iff \varepsilon^* \leq 0\), where \(\varepsilon^* = \max \varepsilon\), subject to the constraints \((a'), (b), (c'), (d), (e), (f')\), plus the constraint \((u_i(w) - u_i(z)) - (u_i(x) - u_i(y)) \geq \varepsilon\).

6.5. GRIP decision support process

GRIP decision support process is composed of five main levels shown in Fig. 1.

Level 1 concerns the input data, i.e., the consistent family of criteria \(F\), and the set of actions \(A\). In addition to the actions to be ranked by GRIP, set \(A\) can also contain some fictitious, past or other auxiliary actions, which will enter the set of reference actions \(A^R\) in order to facilitate elicitation of preference information by the DM.

Level 2 is related to the preference information provided by the DM. The set of reference or training actions \(A^R\) is defined with the help of the DM. The major piece of information provided by the DM is a partial preorder on \(A^R\), which is composed of holistic pairwise comparisons of actions from \(A^R\), and holistic and/or partial preference information on intensities of preferences for some pairs of actions from \(A^R\). It is worth noting that GRIP can easily handle other kinds
of preference information, like local tradeoffs, e.g., the tradeoff information: “the increase of \( g(x) = x \) to \( g(x') = x' \), \( x' > x \), is at least as attractive as the decrease of \( g(x) = \beta \) to \( g(x') = \beta' \), \( \beta' < \beta \), for criteria \( i, j \in I' \), can be handled in the LP constraint \( u_i(x') - u_i(x) \geq u_i(\beta) - u_i(\beta') \). For example, “the increase of \( g(x) = 6 \) by 3 is at least as attractive as the decrease of \( g(x) = 8 \) by 2”, is represented by the LP constraint \( u_i(9) - u_i(6) \geq u_i(8) - u_i(6) \).

In Level 3, the preference information provided by the DM is formally represented by constraints (a) to (i) in Section 6.3.1.

Level 4 concerns the computation phase, where the procedure should check for the existence of at least one value function compatible with the preference information provided by the DM. If there is no such a value function, then the DM is supported to revise his/her preference information.

When, the preference information is consistent, i.e., there exists at least one value function compatible with such information, in Level 5, the method is producing the following output results (see point (2) in Section 6.3.1):

- a necessary ranking \( \succ_N \), for all pairs of actions \( (x, y) \in A \times A \);
- a possible ranking \( \succ_P \), for all pairs of actions \( (x, y) \in A \times A \);
- a necessary ranking \( \succ_N \), with respect to the comprehensive intensities of preferences for all \( (x, y), (w, z) \in A \times A \times A \);
- a possible ranking \( \succ_P \), with respect to the comprehensive intensities of preferences for all \( (x, y), (w, z) \in A \times A \times A \);
- a necessary ranking \( \succ_N \), with respect to the partial intensities of preferences for all \( (x, y), (w, z) \in A \times A \times A \) and for all criteria \( g_i, i \in I' \);
- a possible ranking \( \succ_P \), with respect to the partial intensities of preferences for all \( (x, y), (w, z) \in A \times A \times A \) and for all criteria \( g_i, i \in I' \).

Of course in practice, there is no need to compute all the above results. Indeed, the most useful output information is provided by the first two necessary and possible rankings \( \succ_N \) and \( \succ_P \). Other results can be computed on request concerning particular couples of pairs of actions.

If the DM feels comfortable and agrees on the conclusions, GRIP stops; otherwise, preference information and/or the input data should be revised.

The GRIP interaction scheme generalizes UTA, UTA\(^{GMS}\) and, in a certain sense, MACBETH interaction schemes. Indeed, in case of using only the information on the intensities of preferences, and checking if there exists at least one compatible additive value function, we obtain similar results to MACBETH. We do not need, however, to determine the weights as MACBETH does.

6.6. A theoretical comparison of GRIP with the analytical hierarchy process

In AHP (Saaty, 1980, 2005), criteria should be compared pairwise with respect to their importance. Actions are also compared pairwise on particular criteria with respect to intensity of preference. The following nine point scale of preference is used: 1-equivalent importance, 3-moderate importance, 5-strong importance, 7-very strong or demonstrated importance, and 9-extreme importance. 2, 4, 6 and 8 are intermediate values between the two adjacent judgements. The intensity of importance of criterion \( g_i \) over criterion \( g_j \) is the inverse of the intensity of importance of \( g_j \) over \( g_i \). Analogously, the intensity of preference of action \( x \) over action \( y \) is the inverse of the intensity of preference of \( y \) over \( x \). The above scale is a ratio scale.

Therefore, the intensity of preference is read as the ratio of weights \( w_i \) and \( w_j \) corresponding to criteria \( g_i \) and \( g_j \), and the intensity of preference is read as the ratio of the attractiveness of \( x \) and the attractiveness of \( y \), with respect to the considered criterion \( g_i \), in terms of value functions, the intensity of preference can be interpreted as the ratio \( u_i(g_i(x)) \cdot \frac{w_i}{w_j} \) and \( u_i(g_i(y)) \cdot \frac{w_j}{w_i} \) of the marginal value functions \( u_i(g_i(x)) \) and \( u_i(g_i(y)) \) from ratio \( \frac{w_i}{w_j} \). Thus, the problem is how to obtain values of \( w_i \) and \( w_j \) from ratio \( \frac{w_i}{w_j} \) and values of \( u_i(g_i(x)) \) and \( u_i(g_i(y)) \) from ratio \( \frac{w_i}{w_j} \).

In AHP, it is proposed that these values are supplied by principal eigenvectors of matrices composed of the ratios \( \frac{w_i}{w_j} \) and \( \frac{w_j}{w_i} \). The marginal value functions \( u_i(g_i(x)) \) are then aggregated by means of a weighted-sum using the weights \( w_i \).

Comparing AHP with GRIP, we can say that with respect to a single criterion, the type of questions addressed to the DM is the same: express the intensity of preference in qualitative-ordinal terms (equal, moderate, strong, very strong, extreme). However, differently from GRIP, this intensity of preference is translated into quantitative terms (the scale from 1 to 9) in a quite arbitrary way. In GRIP, instead, the marginal value functions are just a numerical representation of the original qualitative-ordinal information, and no intermediate transformation into quantitative terms is exogenously imposed.

Other differences between AHP and GRIP are related to the following aspects.

1. In GRIP, the value functions \( u_i(g_i(x)) \) depend mainly on holistic judgements, i.e. comprehensive preferences involving jointly all the criteria, while this is not the case in AHP.
(2) In AHP, the weights $w_i$ of criteria $g_i$ are calculated on the basis of pairwise comparisons of criteria with respect to their importance; in GRIP, this is not the case, because the value functions $u_i(g_i(x))$ are expressed on the same scale and thus they can be summed up without any further weighting.

(3) In AHP, all non-ordered pairs of actions must be compared from the viewpoint of the intensity of preference with respect to each particular criterion. Therefore, if $n$ is the number of actions, and $m$ the number of criteria, then the DM has to answer $m \times \frac{n \times (n-1)}{2}$ questions. Moreover, the DM has to answer questions relative to $n^2$ pairwise comparisons of considered criteria with respect to their importance. This is not the case in GRIP, which accepts partial information about preferences in terms of pairwise comparison of some reference actions. Finally, in GRIP there is no question about comparison of relative importance of criteria.

As far as point (2) is concerned, observe that the weights $w_i$ used in AHP represent tradeoffs between evaluations on different criteria. For this reason it is doubtful if they could be inferred from answers to questions concerning comparison of importance. Therefore, AHP has a problem with meaningfulness of its output with respect to its input, and this is not the case of GRIP.

### 6.7. A theoretical comparison with MACBETH

MACBETH (Measuring Attractiveness by a Categorical Based Evaluation Technique) (Bana e Costa and Vansnick, 1994; Bana e Costa et al., 2005) is a method for multiple criteria decision analysis that appeared in the early nineties. This approach requires from the DM qualitative judgements about differences of value to quantify the relative attractiveness of actions or criteria.

When using MACBETH, the DM is asked to provide preference information about two actions of $A$ at a time, through a qualitative judgement of the difference of attractiveness between them. Seven semantic categories of difference of attractiveness are considered: null, very weak, weak, moderate, strong, very strong, and extreme.

The main idea of MACBETH is to build an interval scale from the preference information provided by the DM. It is, however, necessary that the above categories correspond to disjoint intervals (represented in terms of the real numbers). The bounds for such intervals are not arbitrarily fixed a priori, but they are calculated so as to be compatible with the numerical values of all particular actions from $A$, and to ensure compatibility between these values (see Bana e Costa et al., 2005). Linear programming models are used for these calculations. In case of inconsistent judgments, MACBETH provides the DM with information permitting to eliminate such inconsistency.

When comparing MACBETH with GRIP, the following aspects should be considered:

- both deal with qualitative judgements;
- both need a set of comparisons of actions or pairs of actions to work out a numerical representation of preferences, however, MACBETH depends on the specification of two characteristic levels on the original scale, “neutral” and “good”, to obtain the numerical representation of preferences, while GRIP does not need this information;
- GRIP adopts the “disaggregation-aggregation” approach and, therefore, it considers mainly holistic judgements relative to comparisons involving jointly all the criteria, which is not the case of MACBETH;
- GRIP is more general than MACBETH since it can take into account the same kind of qualitative judgments as MACBETH (the difference of attractiveness between pairs of actions) and, moreover, the intensity of preferences of the type “$x$ is preferred to $y$ at least as much as $z$ is preferred to $w$”.

As for the last item, it should be noticed that the intensity of preference considered in MACBETH and the intensity coming from comparisons of the type “$x$ is preferred to $y$ at least as strongly as $w$ is preferred to $z$” (i.e., the quaternary relation $\succsim$) are substantially the same. In fact, the intensities of preference are equivalence classes of the preorder generated by $\succsim^*$. This means that all the pairs $(x,y)$ and $(w,z)$, such that $x$ is preferred to $y$ with the same intensity as $w$ is preferred to $z$, belong to the same semantic category of difference of attractiveness considered in MACBETH. To be more precise, the structure of intensity of preference considered in MACBETH is a particular case of the structure of intensity of preference represented by $\succsim^*$ in GRIP. Still more precisely, GRIP has the same structure of intensity as MACBETH when $\succsim^*$ is a complete preorder. When this does not occur, MACBETH cannot be used while GRIP can naturally deal with this situation. If in MACBETH some judgments of intensity of preference are not provided by the DM, then a feasible value of the qualitative judgment is automatically introduced to each missing place by transitivity.

A detailed comparison of GRIP and MACBETH should take into account the following features,
(1) Preference information:
(a) **GRIP**
   (i) Ordinal comprehensive preference information on pairwise comparison of some reference actions, \( x \succeq y \).
   (ii) Absolute qualitative judgements of intensity of preference for some pairs of reference actions (partial and/or comprehensive), e.g. very weak, weak, moderate, etc., extreme intensity of preferences for \( (x, y) \), or
   (iii) comparison of intensities of preference for some pairs of reference actions (partial and/or comprehensive), \( (x, y) \succeq^* (w, z) \) and/or \( (x, y) \asymp^* (w, z) \).
(b) **MACBETH**
   (i) Ordinal preference information with respect to each criterion for all not equally attractive pairs of actions: \( x \succ y \) or \( y \succ x \).
   (ii) Absolute qualitative judgements of differences of attractiveness for all not equally attractive pairs of actions with respect to each criterion, including “good” and “neutral” reference points, e.g. very weak, weak, moderate, etc., extreme intensity of preference.
   (iii) Ordinal preference information for all not equally attractive criteria: \( g_i \) is more important than \( g_j \), or \( g_j \) is more important than \( g_i \).
   (iv) Absolute qualitative judgements of differences of attractiveness for all not equally attractive pairs of criteria, e.g. very weak, weak, moderate, etc., extreme difference of importance.

(2) Preference model and final results:
(a) **GRIP**
   (i) Uses linear programming to identify a set of all additive value functions defined on an interval scale, compatible with preference information.
   (ii) Builds necessary and possible weak preference relations on set \( A \),
       \[ \succ_N \] (partial preorder);
       \[ \succ_P \] (strongly complete and negatively transitive).
   (iii) Builds necessary and possible comprehensive intensity of preference relations on set \( A \times A \),
       \[ \succ^C_N \] (partial preorder);
       \[ \succ^C_P \] (strongly complete and negatively transitive).
   (iv) Builds necessary and possible partial intensity of preference relations on set \( A \times A \),
       \[ \succ^P_N \] (partial preorder);
       \[ \succ^P_P \] (strongly complete and negatively transitive).
(b) **MACBETH**
   (i) Uses linear programming to build a single interval scale for each criterion, compatible with preference information, and computes a numerical marginal value for each action on each criterion.
   (ii) Computes a weight for each criterion.
   (iii) Builds a weighted-sum model on marginal values which is additive piecewise linear or discrete.
   (iv) Uses the model to set up a complete preorder on set \( A \).

(3) Summary of the crucial differences between the two methodologies:
(a) **GRIP** is using preference information relative to: (1) comprehensive preference on a subset of reference actions with respect to all criteria, (2) partial intensity of preference on some single criteria, and (3) comprehensive intensity of preference with respect to all criteria, while **MACBETH** requires preference information on all pairs of actions with respect to each one of the considered criteria.
(b) Information about partial intensity of preference is of the same nature in **GRIP** and **MACBETH** (equivalence classes of relation \( \succeq^i \) correspond to qualitative judgements of **MACBETH**), but in **GRIP** it may not be complete.
(c) **GRIP** is a “disaggregation-aggregation” approach while **MACBETH** makes use of the “aggregation” approach and, therefore, it needs weights to aggregate evaluations on the criteria.
(d) **GRIP** works with all compatible value functions, while **MACBETH** builds a single interval scale for each criterion, even if many such scales would be compatible with preference information.
(e) Distinguishing necessary and possible consequences of using all value functions compatible with preference information, **GRIP** includes a kind of robustness analysis instead of using a single “best-fit” value function.
(f) The necessary and possible preference relations considered in **GRIP** have several properties of general interest for MCDA.
6.8. Other characteristics of GRIP

It is interesting to note the following characteristics of GRIP:

(1) In the absence of any pairwise comparison of reference actions, the preference relation $\succ_N$ boils down to the weak dominance relation $\prec$ on $A \ (x \prec y \iff y_i \geq x_i$, $i \in I$) (Greco et al., in press).

(2) Each pairwise comparison of the type $x \succeq y$, $x, y \in A$, provided by the DM, contributes to enrich $\succ_N$, i.e., if $\succ_N(x, y)^+$ and $\succ_N(x, y)^-$ are preference relations $\succ_N$ with and without information $x \succeq y$, respectively, we have that $\succ_N(x, y)^+ \supseteq \succ_N(x, y)^-$ (Greco et al., in press).

(3) In the absence of any pairwise comparison of pairs of reference actions, the preference relation $\succ^k$ boils down to the weak dominance relation with respect to difference of preferences $\prec'$ on $A \ ((x, y) \prec' (w, x) \iff x_i > w_i$ and $y_i \leq z_i$, for all $i \in I$).

(4) Each pairwise comparison of the type $(x, y) \succ^* (w, z)$, $x, y, w, z \in A$, provided by the DM, contributes to enrich $\succ^k$, i.e., if $\succ^k(x, y, w, z)^+$ and $\succ^k(x, y, w, z)^-$ are preference relations $\succ^k$ with and without information $(x, y) \succ^* (w, z)$, respectively, we have that

$\succ^k(x, y, w, z)^+ \supseteq \succ^k(x, y, w, z)^-$.

(5) In the absence of any pairwise comparison of reference actions, the preference relation $\succ^p$ is a complete relation such that for any pair $x, y \in A$,

$$(x \succ^p y \text{ and } y \succ^p x) \iff \{\text{not } (x \prec y) \text{ and not } (y \prec x)\} \text{ or } (x \prec y \text{ and } y \prec x)$$

(Greco et al., in press).

(6) Each pairwise comparison $x \succeq y$, $x, y \in A$, provided by the DM contributes to impoverish $\succ^p$, i.e., if $\succ^p(x, y)^+$ and $\succ^p(x, y)^-$ are preference relations $\succ^p$ with and without information $x \succeq y$, respectively, we have that $\succ^p(x, y)^+ \subseteq \succ^p(x, y)^-$ (Greco et al., in press).

(7) In the absence of any pairwise comparison of pairs of two reference actions, the preference relation $\succ^p$ is a complete relation such that for any pair $x, y, w, z \in A$,

$$(x, y) \succ^p (w, z) \text{ and } (w, z) \succ^p (x, y) \iff \{\text{not } [(x, y) \prec' (w, z)] \text{ and not } [(w, z) \prec' (x, y)]\} \text{ or } ((w, z) \prec' (w, z) \text{ and } (w, z) \prec' (x, y)),$$

$$(x, y) \succ^p (c, d) \text{ and not } [(w, z) \succ^p (x, y)] \iff ((x, y) \prec' (w, z) \text{ and } not [(w, z) \prec' (x, y)]).$$

(8) Each pairwise comparison of the type $(x, y) \succ^* (w, z)$, $x, y, w, z \in A$, provided by the DM contributes to impoverish $\succ^p$, i.e., if $\succ^p(x, y, w, z)^+$ and $\succ^p(x, y, w, z)^-$ are preference relations $\succ^p$ with and without information $(x, y) \succ^* (w, z)$, respectively, we have that

$\succ^p(x, y, w, z)^+ \subseteq \succ^p(x, y, w, z)^-$.

7. Conclusion and directions for future research

In this paper, we proposed the GRIP methodology for building a set of additive value functions compatible with the following preference information provided by the DM: a partial preorder in the set of reference actions and/or partial and comprehensive comparisons of intensities of preference between some pairs of reference actions. This preference information is used within a newly designed ordinal regression approach.

Considering all the compatible value functions, GRIP permits to achieve the following results:

- a necessary weak preference relation $\succ_N$ on $A$, being a partial preorder,
- a possible weak preference relation $\succ^p$ on $A$, being a strongly complete and negatively transitive relation,
- a comprehensive necessary intensity of preference relation $\succ_N^k$, being a partial preorder on $A \times A$,
- a comprehensive possible intensity of preference relation $\succ^p$, being a strongly complete and negatively transitive relation on $A \times A$,
- a partial necessary intensity of preference relation $\succ_N^k$, being a partial preorder on $A \times A$,
- a partial possible intensity of preference relation $\succ^p$, being a strongly complete and negatively transitive relation on $A \times A$. 

There are several topics which constitute interesting directions for future research, in particular:

(1) Handling preference information with gradual credibility.
(2) Building a decision support system based on the GRIP methodology.
(3) Using GRIP for interactive Multi-Objective Optimization problems (for a first result see Figueira et al., 2008).
(4) Extending the method to group decision-making situations.
(5) Applying the method to real-world problems.

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Appendix

Proof of Theorem 5.1. In what follows $\mathcal{U}$ represents the set of compatible value functions satisfying constraints from (a)–(i).

(1) For all $x, y \in A$,
$$x \trianglerighteq^N y \iff \forall U \in \mathcal{U}, U(x) \geq U(y) \Rightarrow \exists U \in \mathcal{U} \text{ such that } U(x) \geq U(y) \iff x \trianglerighteq^p y.$$  

(2) For all $x, y \in A^p$,
$$x \trianglerighteq y \Rightarrow \forall U \in \mathcal{U}, U(x) \geq U(y) \iff x \trianglerighteq^N y.$$  

(3) Let us recall that a partial preorder is a transitive and reflexive binary relation. Thus, in order to show that $\trianglerighteq^N$ is a partial preorder, we have to show that it is transitive and reflexive,
- $\trianglerighteq^N$ is transitive: for all $x, y, z \in A$,
  $$x \trianglerighteq^N y \text{ and } y \trianglerighteq^N z \iff \forall U \in \mathcal{U}, U(x) \geq U(y) \text{ and } U(y) \geq U(z) \Rightarrow \forall U \in \mathcal{U}, U(x) \geq U(z) \Rightarrow x \trianglerighteq^N z.$$  
- $\trianglerighteq^N$ is reflexive: for all $x \in A$,
  $$U(x) = U(x), \forall U \in \mathcal{U} \iff U(x) \geq U(x), \forall U \in \mathcal{U} \iff x \trianglerighteq^N x.$$ Thus, we proved that $\trianglerighteq^N$ is a partial preorder.

$\trianglerighteq^p$ is strongly complete: for all $x, y \in A$,
$$U(x) \geq U(y) \text{ or } U(y) \geq U(x), \forall U \in \mathcal{U} \iff x \trianglerighteq^p y \text{ or } y \trianglerighteq^p x.$$  

$\trianglerighteq^p$ is negatively transitive: for all $x, y, z \in A$,
$$\text{not (} x \trianglerighteq^p y \text{) and not (} y \trianglerighteq^p z \text{)} \iff \not \exists U \in \mathcal{U}, \text{ such that } U(x) \geq U(y), \text{ and } \not \exists U \in \mathcal{U}, \text{ such that } U(y) \geq U(z) \iff \forall U \in \mathcal{U}, U(x) < U(y) \text{ and } U(y) < U(z) \Rightarrow \forall U \in \mathcal{U}, U(x) < U(z) \iff \not \exists U \in \mathcal{U}, \text{ such that } U(x) \geq U(z) \iff \not \text{ (} x \trianglerighteq^p z \text{).}$$  

(4) For all $x, y \in A$,
$$U(x) \geq U(y) \forall U \in \mathcal{U} \text{ or } \exists U \in \mathcal{U} \text{ such that } U(y) > U(x) \Rightarrow x \trianglerighteq^N y \text{ or } y \trianglerighteq^p x.$$  

(5) For all $x, y, z \in A$,
$$x \trianglerighteq^N y \text{ and } y \trianglerighteq^p z \iff U(x) \geq U(y) \forall U \in \mathcal{U} \text{ and } U(y) \geq U(z) \text{ for at least one } U \in \mathcal{U} \Rightarrow U(x) \geq U(z) \text{ for at least one } U \in \mathcal{U} \iff x \trianglerighteq^p z.$$  

\( \trianglerighteq^N \) represents the set of compatible value functions satisfying constraints from (a)–(i).
(6) For all \( x, y, z \in A \),
\[ x \succeq^N y \text{ and } y \succeq^N z \iff U(x) \geq U(y) \text{ for at least one } U \in \mathcal{U} \text{ and } U(y) \geq U(z) \forall U \in \mathcal{U} \]
\[ \Rightarrow U(x) \geq U(z) \text{ for at least one } U \in \mathcal{U} \iff x \succeq^N z. \]

(7) For all \( x, y, w, z \in A^k \),
\[ (x, y) \succeq^N (w, z) \iff U(x) - U(y) \geq U(w) - U(z) \forall U \in \mathcal{U} \]
\[ \Rightarrow U(x) - U(y) \geq U(w) - U(z) \text{ for at least one } U \in \mathcal{U} \iff (x, y) \succeq^N (w, z). \]

(8) For all \( x, y, w, z \in A^k \),
\[ (x, y) \succeq^N (w, z) \Rightarrow U(x) - U(y) \geq U(w) - U(z) \forall U \in \mathcal{U} \iff (x, y) \succeq^N (w, z). \]

(9) To show that \( \succeq^N \) is a partial preorder we have to show that it is transitive and reflexive,

• \( \succeq^N \) is transitive: for all \( x, y, w, z, r, s \in A \),
\[ (x, y) \succeq^N (w, z) \text{ and } (w, z) \succeq^N (r, s) \]
\[ \iff U(x) - U(y) \geq U(w) - U(z) \text{ and } U(w) - U(z) \geq U(r) - U(s) \forall U \in \mathcal{U} \]
\[ \Rightarrow U(x) - U(y) \geq U(r) - U(s) \forall U \in \mathcal{U} \iff (x, y) \succeq^N (r, s). \]

• \( \succeq^N \) is reflexive: for all \( x, y \in A \),
\[ U(x) - U(y) = U(x) - U(y) \forall U \in \mathcal{U} \]
\[ \iff U(x) - U(y) \geq U(x) - U(y) \forall U \in \mathcal{U} \iff (x, y) \succeq^N (x, y). \]

Thus we proved that \( \succeq^N \) is a partial preorder.

\( \succeq^N \) is strongly complete: for all \( x, y, w, z \in A \),
\[ U(x) - U(y) \geq U(w) - U(z) \text{ or } U(w) - U(z) \geq U(x) - U(y) \forall U \in \mathcal{U} \]
\[ \iff (x, y) \succeq^N (w, z) \text{ or } (w, z) \succeq^N (x, y). \]

\( \succeq^N \) is negatively transitive: for all \( x, y, w, z, r, s \in A \),
\[ \text{not } (x, y) \succeq^N (w, z) \text{ and not } (w, z) \succeq^N (r, s) \]
\[ \iff \nexists \ U \in \mathcal{U} \text{ such that } [U(x) - U(y)] \geq [U(w) - U(z)] \text{ and } \]
\[ \nexists \ U \in \mathcal{U} \text{ such that } [U(w) - U(z)] \geq [U(r) - U(s)] \]
\[ \iff \forall U \in \mathcal{U}, [U(x) - U(y)] < [U(w) - U(z)] \text{ and } [U(w) - U(z)] < [U(r) - U(s)] \]
\[ \Rightarrow \forall U \in \mathcal{U}, [U(x) - U(y)] < [U(r) - U(s)] \]
\[ \iff \nexists \ U \in \mathcal{U} \text{ such that } [U(x) - U(y)] \geq [U(r) - U(s)] \]
\[ \iff \text{not } (x, y) \succeq^N (r, s). \]

(10) For all \( x, y, w, z \in A \),
\[ U(x) - U(y) \geq U(w) - U(z) \forall U \in \mathcal{U} \text{ or } \exists U \in \mathcal{U} \text{ such that } U(w) - U(z) > U(x) - U(y) \]
\[ \Rightarrow (x, y) \succeq^N (w, z) \text{ or } (w, z) \succeq^N (x, y). \]

(11) For all \( x, y, w, z, r, s \in A \),
\[ (x, y) \succeq^N (w, z) \text{ and } (w, z) \succeq^N (r, s) \]
\[ \iff U(x) - U(y) \geq U(w) - U(z) \forall U \in \mathcal{U} \text{ and } \]
\[ U(w) - U(z) \geq U(r) - U(s) \text{ for at least one } U \in \mathcal{U} \]
\[ \Rightarrow U(x) - U(y) \geq U(r) - U(s) \text{ for at least one } U \in \mathcal{U} \]
\[ \iff (x, y) \succeq^N (r, s). \]
For all \(x, y, w, z, r, s \in A\),
\[(x, y) \succeq^y (w, z) \text{ and } (w, z) \succeq^x (r, s)\]
\[\iff U(x) - U(y) \geq U(w) - U(z) \text{ for at least one } U \in \mathcal{U} \text{ and }\]
\[U(w) - U(z) \geq U(r) - U(s) \forall U \in \mathcal{U}\]
\[\Rightarrow U(x) - U(y) \geq U(r) - U(s) \forall U \in \mathcal{U}\]
\[\iff (x, y) \succeq^r (r, s).\]

For all \(x, x', y, w, z \in A\),
\[x' \succeq^y x \text{ and } (x, y) \succeq^z (w, z)\]
\[\iff U(x') \geq U(x) \forall U \in \mathcal{U} \text{ and }\]
\[U(x) - U(y) \geq [U(w) - U(z)] \forall U \in \mathcal{U}\]
\[\Rightarrow [U(x') - U(y)] \geq [U(w) - U(z)] \forall U \in \mathcal{U}\]
\[\iff (x', y) \succeq^z (w, z).\]

For all \(x, x', y, w, z \in A\),
\[x' \succeq^y x \text{ and } (x, y) \succeq^z (w, z)\]
\[\iff U(x') \geq U(x) \forall U \in \mathcal{U} \text{ and }\]
\[U(x) - U(y) \geq [U(w) - U(z)] \forall U \in \mathcal{U}\]
\[\Rightarrow [U(x') - U(y)] \geq [U(w) - U(z)] \forall U \in \mathcal{U}\]
\[\iff (x', y) \succeq^z (w, z).\]

For all \(x, y, y', w, z \in A\),
\[y \succeq^y y' \text{ and } (x, y) \succeq^z (w, z)\]
\[\iff U(y) \geq U(y') \forall U \in \mathcal{U} \text{ and } [U(x) - U(y)] \geq [U(w) - U(z)] \forall U \in \mathcal{U}\]
\[\Rightarrow [U(x) - U(y')] \geq [U(w) - U(z)] \forall U \in \mathcal{U}\]
\[\iff (x, y') \succeq^z (w, z).\]

For all \(x, y, y', w, z \in A\),
\[y \succeq^y y' \text{ and } (x, y) \succeq^z (w, z)\]
\[\iff U(y) \geq U(y') \forall U \in \mathcal{U} \text{ and } [U(x) - U(y)] \geq [U(w) - U(z)] \forall U \in \mathcal{U}\]
\[\Rightarrow [U(x) - U(y')] \geq [U(w) - U(z)] \forall U \in \mathcal{U}\]
\[\iff (x, y') \succeq^z (w, z).\]
(19) For all \(x, y, w, w', z \in A\),
\[
\begin{align*}
&w \succsim^N w' \text{ and } (x, y) \succsim^\sigma(w, z) \\
\iff& U(w) \geq U(w') \text{ and } [U(x) - U(y)] \geq [U(w) - U(z)] \forall U \in \mathcal{U} \\
\Rightarrow& (x, y) \succsim^\sigma(w', z).
\end{align*}
\]
(20) For all \(x, y, w, w', z \in A\),
\[
\begin{align*}
&w \succsim^N w' \text{ and } (x, y) \succsim^\sigma(w, z) \\
\iff& U(w) \geq U(w') \forall U \in \mathcal{U} \text{ and } [U(x) - U(y)] \geq [U(w) - U(z)] \\
\text{for at least one } & U \in \mathcal{U} \\
\Rightarrow& [U(x) - U(y)] \geq [U(w') - U(z)] \text{ for at least one } U \in \mathcal{U} \\
\iff& (x, y) \succsim^\sigma(w', z).
\end{align*}
\]
(21) For all \(x, y, w, w', z \in A\),
\[
\begin{align*}
&w \succsim^P w' \text{ and } (x, y) \succsim^\sigma(w, z) \\
\iff& U(w) \geq U(w') \text{ for at least one } U \in \mathcal{U} \text{ and} \\
&[U(x) - U(y)] \geq [U(w) - U(z)] \forall U \in \mathcal{U} \\
\Rightarrow& [U(x) - U(y)] \geq [U(w') - U(z)] \text{ for at least one } U \in \mathcal{U} \\
\iff& (x, y) \succsim^\sigma(w', z).
\end{align*}
\]
(22) For all \(x, y, w, z, z' \in A\),
\[
\begin{align*}
&z' \succ ^N z \text{ and } (x, y) \succsim ^\sigma(w, z) \\
\iff& U(z') \geq U(z) \text{ and } [U(x) - U(y)] \geq [U(w) - U(z)] \forall U \in \mathcal{U} \\
\Rightarrow& [U(x) - U(y)] \geq [U(w') - U(z')] \text{ for at least one } U \in \mathcal{U} \\
\iff& (x, y) \succsim ^\sigma(w', z').
\end{align*}
\]
(23) For all \(x, y, w, z, z' \in A\),
\[
\begin{align*}
&z' \succ ^N z \text{ and } (x, y) \succsim ^\sigma(w, z) \\
\iff& U(z') \geq U(z) \forall U \in \mathcal{U} \text{ and } [U(x) - U(y)] \geq [U(w) - U(z)] \\
\text{for at least one } & U \in \mathcal{U} \\
\Rightarrow& [U(x) - U(y)] \geq [U(w) - U(z')] \text{ for at least one } U \in \mathcal{U} \\
\iff& (x, y) \succsim ^\sigma(w', z').
\end{align*}
\]
(24) For all \(x, y, w, z, z' \in A\),
\[
\begin{align*}
&z' \succ ^P z \text{ and } (x, y) \succsim ^\sigma(w, z) \\
\iff& U(z') \geq U(z) \text{ for at least one } U \in \mathcal{U} \text{ and} \\
&[U(x) - U(y)] \geq [U(w) - U(z)] \forall U \in \mathcal{U} \\
\Rightarrow& [U(x) - U(y)] \geq [U(w) - U(z')] \text{ for at least one } U \in \mathcal{U} \\
\iff& (x, y) \succsim ^\sigma(w', z').
\end{align*}
\]
(25) For all \(x, x', y \in A\),
\[
\begin{align*}
&(x', y) \succsim ^w(x, y) \\
\iff& [U(x') - U(y)] \geq [U(x) - U(y)] \forall U \in \mathcal{U} \\
\iff& U(x') \geq U(x) \forall U \in \mathcal{U} \\
\iff& x' \succsim ^N x.
\end{align*}
\]
(26) For all \(x, x', y \in A\),
\[(x', y) \succeq^x (x, y) \iff [U(x') - U(y)] \geq [U(x) - U(y)] \text{ for at least one } U \in \mathcal{U}\]
\[\iff U(x') \geq U(x) \text{ for at least one } U \in \mathcal{U}\]
\[\iff x' \succeq^x x.\]

(27) For all \(x, y, y' \in A\),
\[(x, y) \succeq^y (x, y') \iff [U(x) - U(y)] \geq [U(x) - U(y')] \forall U \in \mathcal{U}\]
\[\iff U(y') \geq U(y) \forall U \in \mathcal{U}\]
\[\iff y' \succeq^y y.\]

(28) For all \(x, y, y' \in A\),
\[(x, y) \succeq^r (x, y') \iff [U(x) - U(y)] \geq [U(x) - U(y')] \forall U \in \mathcal{U}\]
\[\iff U(y') \geq U(y) \forall U \in \mathcal{U}\]
\[\iff y' \succeq^r y.\]

(29) To show that \(\succeq^x_i\), \(i \in I\), is a partial preorder we have to show that it is transitive and reflexive,

- \(\succeq^y_i\) is transitive: for all \(x, y, z, r, s \in A\),
\[(x, y) \succeq^y_i (w, z) \text{ and } (w, z) \succeq^y_i (r, s) \iff [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \text{ and }\]
\[[u_i(g_i(w)) - u_i(g_i(z))] \geq [u_i(g_i(r)) - u_i(g_i(s))] \forall U \in \mathcal{U}\]
\[\Rightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(r)) - u_i(g_i(s))] \forall U \in \mathcal{U}\]
\[\iff (x, y) \succeq^y_i (r, s).\]

- \(\succeq^y_i\) is reflexive: for all \(x, y \in A\),
\[[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(x)) - u_i(g_i(y))] \forall U \in \mathcal{U}\]
\[\iff (x, y) \succeq^y_i (x, y).\]

Thus we proved that \(\succeq^x_i\), \(i \in I\), is a partial preorder.

\(\succeq^x_i\), \(i \in I\), is strongly complete: for all \(x, y, z, A \in \mathcal{U}\),
\[[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \text{ or }\]
\[[u_i(g_i(w)) - u_i(g_i(z))] \geq [u_i(g_i(x)) - u_i(g_i(y))] \forall U \in \mathcal{U}\]
\[\iff (x, y) \succeq^x_i (w, z) \text{ or } (w, z) \succeq^x_i (x, y).\]

\(\succeq^x_i\), \(i \in I\), is negatively transitive: for all \(x, y, w, z, r, s \in A\),
\[\text{not } (x, y) \succeq^r_i (w, z) \text{ and not } (w, z) \succeq^r_i (r, s) \iff \exists U \in \mathcal{U} \text{ such that } [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \text{ and }\]
\[\exists U \in \mathcal{U} \text{ such that } [u_i(g_i(w)) - u_i(g_i(z))] \geq [u_i(g_i(r)) - u_i(g_i(s))] \forall U \in \mathcal{U}\]
\[\Rightarrow [u_i(g_i(x)) - u_i(g_i(y))] 
< [u_i(g_i(w)) - u_i(g_i(z))] \text{ and }\]
\[[u_i(g_i(w)) - u_i(g_i(z))] < [u_i(g_i(r)) - u_i(g_i(s))] \forall U \in \mathcal{U}\]
\[\Rightarrow \exists U \in \mathcal{U} \text{ such that } [u_i(g_i(x)) - u_i(g_i(y))] 
\geq [u_i(g_i(r)) - u_i(g_i(s))] \text{ and }\]
\[\text{not } (x, y) \succeq^r_i (r, s).\]
(30) For all \( x, y, w, z \in A, \ i \in I \),
\[ u_i (x) - u_i (y) \geq u_i (w) - u_i (z) \ \forall U \in \mathcal{U} \ or \ \exists U \in \mathcal{U} \] such that \( u_i (w) - u_i (z) > u_i (x) - u_i (y) \)
\[ \Rightarrow (x, y) \succeq^w_{i} (w, z) \ or \ (w, z) \succeq^w_{i} (x, y). \]

(31) For all \( x, y, w, z, r, s \in A, \ i \in I \),
\[ (x, y) \succeq^w_{i} (w, z) \ and \ (w, z) \succeq^w_{i} (r, s) \]
\[ \iff [u_i (g_i (x)) - u_i (g_i (y))] \geq [u_i (g_i (w)) - u_i (g_i (z))] \ \forall U \in \mathcal{U} \ and \]
\[ [u_i (g_i (w)) - u_i (g_i (z))] \geq [u_i (g_i (r)) - u_i (g_i (s))] \ \forall U \in \mathcal{U} \]
\[ \Rightarrow [u_i (g_i (x)) - u_i (g_i (y))] \geq [u_i (g_i (r)) - u_i (g_i (s))] \ \forall U \in \mathcal{U} \]
\[ \iff (x, y) \succeq^w_{i} (r, s). \]

(32) For all \( x, y, w, z, r, s \in A, \ i \in I \),
\[ (x, y) \succeq^w_{i} (w, z) \ and \ (w, z) \succeq^w_{i} (r, s) \]
\[ \iff [u_i (g_i (x)) - u_i (g_i (y))] \geq [u_i (g_i (w)) - u_i (g_i (z))] \ \forall U \in \mathcal{U} \ and \]
\[ [u_i (g_i (w)) - u_i (g_i (z))] \geq [u_i (g_i (r)) - u_i (g_i (s))] \ \forall U \in \mathcal{U} \]
\[ \Rightarrow [u_i (g_i (x)) - u_i (g_i (y))] \geq [u_i (g_i (r)) - u_i (g_i (s))] \ \forall U \in \mathcal{U} \]
\[ \iff (x, y) \succeq^w_{i} (r, s). \]

(33) For all \( x', y, w, z \in A, \ i \in I \),
\[ g_i (x') \geq g_i (x) \ and \ (x, y) \succeq^w_{i} (w, z) \]
\[ \iff u_i (g_i (x')) \geq u_i (g_i (x)) \ and \ [u_i (g_i (x)) - u_i (g_i (y))] \geq [u_i (g_i (w)) - u_i (g_i (z))] \ \forall U \in \mathcal{U} \]
\[ \Rightarrow [u_i (g_i (x')) - u_i (g_i (y))] \geq [u_i (g_i (w)) - u_i (g_i (z))] \ \forall U \in \mathcal{U} \]
\[ \iff (x', y) \succeq^w_{i} (w, z). \]

(34) For all \( x', y, w, z \in A, \ i \in I \),
\[ g_i (x') \geq g_i (x) \ and \ (x, y) \succeq^w_{i} (w, z) \]
\[ \iff u_i (g_i (x')) \geq u_i (g_i (x)) \ and \ [u_i (g_i (x)) - u_i (g_i (y))] \geq [u_i (g_i (w)) - u_i (g_i (z))] \ \forall U \in \mathcal{U} \]
\[ \Rightarrow [u_i (g_i (x')) - u_i (g_i (y))] \geq [u_i (g_i (w)) - u_i (g_i (z))] \ \forall U \in \mathcal{U} \]
\[ \iff (x', y) \succeq^w_{i} (w, z). \]

(35) For all \( x, y, y', w, z \in A, \ i \in I \),
\[ g_i (y) \geq g_i (y') \ and \ (x, y) \succeq^w_{i} (w, z) \]
\[ \iff u_i (g_i (y)) \geq u_i (g_i (y')) \ and \ [u_i (g_i (x)) - u_i (g_i (y))] \geq [u_i (g_i (w)) - u_i (g_i (z))] \ \forall U \in \mathcal{U} \]
\[ \Rightarrow [u_i (g_i (x)) - u_i (g_i (y'))] \geq [u_i (g_i (w)) - u_i (g_i (z))] \ \forall U \in \mathcal{U} \]
\[ \iff (x', y') \succeq^w_{i} (w, z). \]

(36) For all \( x, y, y', w, z \in A, \ i \in I \),
\[ g_i (x) \geq g_i (y) \ and \ (x, y) \succeq^w_{i} (w, z) \]
\[ \iff u_i (g_i (x)) \geq u_i (g_i (y')) \ and \ [u_i (g_i (x)) - u_i (g_i (y))] \geq [u_i (g_i (w)) - u_i (g_i (z))] \ \forall U \in \mathcal{U} \]
\[ \Rightarrow [u_i (g_i (x)) - u_i (g_i (y'))] \geq [u_i (g_i (w)) - u_i (g_i (z))] \ \forall U \in \mathcal{U} \]
\[ \iff (x, y') \succeq^w_{i} (w, z). \]
(37) For all \( x, y, w, w', z \in A, i \in I \),
\[
g_i(w) \geq g_i(w') \text{ and } (x, y) \succsim^w_i (w, z)
\]
\[
\iff u_i(g_i(w)) \geq u_i(g_i(w')) \text{ and }
[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \forall U \in \mathcal{U}
\]
\[
\Rightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w')) - u_i(g_i(z))] \forall U \in \mathcal{U}
\]
\[
\iff (x, y') \succsim^w_i (w', z).
\]

(38) For all \( x, y, w, w', z \in A, i \in I \),
\[
g_i(w) \geq g_i(w') \text{ and } (x, y) \succsim^w_i (w, z)
\]
\[
\iff u_i(g_i(w)) \geq u_i(g_i(w')) \text{ and }
[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \text{ for at least one } U \in \mathcal{U}
\]
\[
\Rightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w')) - u_i(g_i(z))] \text{ for at least one } U \in \mathcal{U}
\]
\[
\iff (x, y) \succsim^w_i (w', z).
\]

(39) For all \( x, y, w, z, z' \in A, i \in I \),
\[
g_i(z') \geq g_i(z) \text{ and } (x, y) \succsim^w_i (w, z)
\]
\[
\iff u_i(g_i(z')) \geq u_i(g_i(z)) \text{ and }
[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \forall U \in \mathcal{U}
\]
\[
\Rightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z'))] \forall U \in \mathcal{U}
\]
\[
\iff (x, y) \succsim^w_i (w, z').
\]

(40) For all \( x, y, w, z, z' \in A, i \in I \),
\[
g_i(z') \geq g_i(z) \text{ and } (x, y) \succsim^w_i (w, z)
\]
\[
\iff u_i(g_i(z')) \geq u_i(g_i(z)) \text{ and }
[u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z))] \text{ for at least one } U \in \mathcal{U}
\]
\[
\Rightarrow [u_i(g_i(x)) - u_i(g_i(y))] \geq [u_i(g_i(w)) - u_i(g_i(z'))] \text{ for at least one } U \in \mathcal{U}
\]
\[
\iff (x, y) \succsim^w_i (w, z').
\]

(41) For all \( x, x', y \in A, i \in I \),
\[
g_i(x') \geq g_i(x) \Rightarrow u_i(g_i(x')) \geq u_i(g_i(x)) \forall U \in \mathcal{U}
\]
\[
\iff [u_i(g_i(x')) - u_i(g_i(y))] \geq [u_i(g_i(x)) - u_i(g_i(y))] \forall U \in \mathcal{U}
\]
\[
\iff (x', y) \succsim^w_i (x, y).
\]

(42) For all \( x, x', y \in A, i \in I \),
\[
(x', y) \succsim^w_i (x, y)
\]
\[
\iff [u_i(g_i(x')) - u_i(g_i(y))] > [u_i(g_i(x)) - u_i(g_i(y))] \text{ for at least one } U \in \mathcal{U}
\]
\[
\iff u_i(g_i(x')) > u_i(g_i(x)) \text{ for at least one } U \in \mathcal{U}
\]
\[
\Rightarrow g_i(x') > g_i(x).
\]

(43) For all \( x, y, y' \in A, i \in I \),
\[
g_i(y) \geq g_i(y') \Rightarrow u_i(g_i(y)) \geq u_i(g_i(y')) \forall U \in \mathcal{U}
\]
\[
\iff [u_i(g_i(x)) - u_i(g_i(y'))] \geq [u_i(g_i(x)) - u_i(g_i(y))] \forall U \in \mathcal{U}
\]
\[
\iff (x, y') \succsim^w_i (x, y).
\]
(44) For all $x, x', y \in A$, $i \in I$,

$$
(x, y) > (x', y') \iff \left[ u_i(g_i(x)) - u_i(g_i(y)) \right] > \left[ u_i(g_i(x)) - u_i(g_i(y')) \right] \text{ for at least one } U \in \mathcal{U}
$$

$$
\iff u_i(g_i(y')) > u_i(g_i(y)) \text{ for at least one } U \in \mathcal{U}
$$

$$
g(y') > g(y). \quad \square
$$

References


